Introduction to Stability Theory of Dynamical Systems

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Chapter 1

Dynamical Systems

1.1 Definitions and Related Notation

Definition 1.1.1. *A **dynamical system** on X is the triplet $(X, \mathbb{R}, \varphi)$, where φ is a map from the product space $X \times \mathbb{R}$ into the space X satisfying the following axioms:*

$$\varphi(x, 0) = x, \forall x \in X \tag{1.1}$$

$$\varphi(\varphi(x, t_1), t_2) = \varphi(x, t_1 + t_2), \forall x \in X \text{ and } t_1, t_2 \in \mathbb{R} \tag{1.2}$$

$$\varphi \text{ is continuous} \tag{1.3}$$

*Given a dynamical system acting on X, the space X is called the **phase space** and the map φ is called the **phase map**. We will delete the symbol φ and denote the image $\varphi(x,t)$ of a point (x,t) in $X \times \mathbb{R}$ as $t.x$. The above identities will then read*

$$0.x = x, \forall x \in X \tag{1.4}$$

$$t_2.(t_1.x) = (t_1 + t_2).x, \forall x \in X \text{ and } t_1, t_2 \in \mathbb{R} \tag{1.5}$$

In accordance to this notation, if $M \subset X$ and $A \subset \mathbb{R}$, then $A.M$ is the set $\{t.x : x \in M \text{ and } t \in A\}$. If either M or A is a singleton, i.e., $M = \{x\}$ or $A = \{t\}$, then we simply write $A.x$ or $t.M$ for $\{x\}A$ and $M\{t\}$, respectively. For any $x \in X$, the set $\mathbb{R}.x$ is called the trajectory through x.

*The phase map determines two other maps when one of the variables x or t is fixed. Thus for fixed $t \in \mathbb{R}$, the map $\varphi^t : X \to X$ defined by $\varphi^t(x) = t.x$ is called a **transition**, and for a fixed $x \in X$, the map $\varphi_x : \mathbb{R} \to X$ defined by $\varphi_x(t) = t.x$ is called a **motion** (through x). Note that φ_x maps $\mathbb{R}$ onto $\mathbb{R}.x$.*

1.2 Examples of Dynamical Systems

Definition 1.2.1. *Ordinary Autonomous Differential Systems. Consider the autonomous differential system*

$$\frac{dx}{dt} = \dot{x} = f(x), \tag{1.6}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$($\mathbb{R}^n$ is the real n-dimensional euclidean space) is continuous and moreover assume that for each $x \in \mathbb{R}^n$ a unique solution $t.x$ exists which is defined on $\mathbb{R}$ and satisfies $0.x = x$. Then it is well known that the uniqueness of solutions implies

$$t_1.(t_2.x) = (t_1 + t_2).x, \ \forall t_1, t_2 \in \mathbb{R}, \tag{1.7}$$

and considered as a function from $\mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}^n, \varphi$ is continuous in its arguments. We remark that the conditions on solutions of 1.6, as required above, are obtained, for example, if the function f satisfies a global Lipschitz condition, i.e., there is a positive number k such that

$$\|f(x) - f(y)\| \leqq k\|x - y\|, \ \forall x, y \in \mathbb{R}^n \tag{1.8}$$

Definition 1.2.2. *Ordinary Non-autonomous Differential Systems. Consider the differential system*

$$\dot{x} = f(t, x), x \in \mathbb{R}^n, t \in \mathbb{R} \tag{1.9}$$

where $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous. Assume that for each $(t_0, x_0) \in \mathbb{R}\times\mathbb{R}^n$, 1.9 *possesses a unique solution $\varphi(t, t_0, x_0), \varphi(t_0, t_0, x_0) = x_0$, defined $\forall t \in \mathbb{R}$.*

$$\dot{y} = g(y), \text{where } y = (t, x), \text{ and } g(y) = (1, f(y)). \tag{1.10}$$

Chapter 2

Elementary Concepts

2.1 Invariant Sets and Trajectories

Definition 2.1.1. *A set $M \subset X$ is called* ***invariant*** *whenever*

$$t.x \in M, \forall x \in M \text{ and } t \in \mathbb{R}. \tag{2.1}$$

It is called ***positively invariant*** *whenever 2.1 holds with $\mathbb{R}$ replaced by $\mathbb{R}^+$, and is called* ***negatively invariant*** *if the same holds with $\mathbb{R}$ replaced by $\mathbb{R}^-$. Note that 2.1 is equivalent to $R.M = M$.*

Theorem 2.1.2. *Let $\{M_i\}$ be a collection of positively invariant, negatively invariant, or invariant subsets of X. Then their intersection and their union have the same property.*

Theorem 2.1.3. *Let $M \subset X$ be positively invariant, negatively invariant, or invariant. Then the closure $\overline{M}$ has the same property.*

Theorem 2.1.4. *A set $M \subset X$ is positively invariant if and only if the set $X \setminus M$ is negatively invariant. M is invariant if and only if $X \setminus M$ is invariant.*

Theorem 2.1.5. *A set $M \subset X$ is invariant if and only if it is both positively and negatively invariant.*

Definition 2.1.6. *We introduce the maps γ, γ^+, and γ^- from X into 2^X by defining $\forall x \in X$,*

$$\gamma(x) = \{t.x : t \in \mathbb{R}\}, \tag{2.2}$$

$$\gamma^+(x) = \{t.x : t \in \mathbb{R}^+\}, \tag{2.3}$$

$$\gamma^-(x) = \{t.x : t \in \mathbb{R}^-\}, \tag{2.4}$$

$\forall x \in X$, the sets $\gamma(x), \gamma^+(x)$, and $\gamma^-(x)$ are, respectively called the ***trajectory****, the* ***positive semi-trajectory****, and the* ***negative semi-trajectory*** *through x. Note that $\forall x \in X, \gamma(x) = \mathbb{R}.x$, etc.*

Proposition 2.1.7. *A set $M \subset X$ is invariant, positively invariant, or negatively invariant if and only if, respectively, $\gamma(M) = M, \gamma^+(M) = M$, or $\gamma^-(M) = M$.*

Another characterization of invariance is the following:

Proposition 2.1.8. *A set $M \subset X$ is invariant, positively invariant, or negatively invariant if and only if for each $x \in M$, respectively, $\gamma(x) \subset M, \gamma^+(x) \subset M, \gamma^-(x) \subset M$.*

2.2 Critical Points and Periodic Points

Definition 2.2.1. *A point $x \in X$ is said to be a critical point if $x = t.x, \forall t \in \mathbb{R}$.*

Theorem 2.2.2. *Let $x \in X$. Then the following are equivalent.*

$$x \text{ is critical}, \tag{2.5}$$

$$\{x\} = \gamma(x), \tag{2.6}$$

$$\{x\} = \gamma^+(x), \tag{2.7}$$

$$\{x\} = \gamma^-(x), \tag{2.8}$$

$$\{x\} = [a, b].x \text{ for some } a<b. \tag{2.9}$$

$$\text{There is a sequence } \{t_n\}, t_n > 0, t_n \to 0 \text{ with } x = t_n.x \text{ for each } n. \tag{2.10}$$

Theorem 2.2.3. *A point $x \in X$ is critical if and only if every neighborhood of x contains a semi-trajectory.*

Definition 2.2.4. *A point $x \in X$ is said to be* ***periodic*** *if there is a $T \neq 0$ such that*

$$t.x = (t + T).x, \forall t \in \mathbb{R}. \tag{2.11}$$

A number $T \in \mathbb{R}$ for which 2.11 holds will be called a period of x. If a point $x \in X$ is periodic then both the motion and the trajectory are said to be periodic. Note that every $x \in X$ has the period $T = 0$, but it may not be periodic. Further, if $x \in X$ is critical, then every $T \in \mathbb{R}$ is a period of x, and indeed x is periodic. The following characterization of a periodic point is very useful.

Proposition 2.2.5. *$x \in X$ is periodic if and only if there is a $T \neq 0$ with $x = T.x$.*

Theorem 2.2.6. *If $x \in X$ is periodic, but not critical, then there is $T > 0$ such that T is the smallest positive period of x. Further if τ is any other period of x, then $\tau = T \cdot n$ for some integer n.*

Definition 2.2.7. *If a point x is periodic but not critical, then the smallest positive period of x is called its fundamental or primitive period.*

2.3 Trajectory Closures and Limit Sets

Definition 2.3.1. *Define maps Λ^+, Λ^- from X into 2^X by setting for each $x \in X$,*

$$\Lambda^+(x) = \{y \in X : \exists \ \{t_n\} \in \mathbb{R} \text{ with } t_n \to +\infty \text{ and } t_n.x \to y\}, \tag{2.12}$$

$$\Lambda^-(x) = \{y \in X : \exists \ \{t_n\} \in \mathbb{R} \text{ with } t_n \to -\infty \text{ and } t_n.x \to y\}, \tag{2.13}$$

For any $x \in X$, the set $\Lambda^+(x)$ is called its ***positive (or omega) limit set****, and the set $\Lambda^-(x)$ is called its* ***negative (or alpha) limit set****.*

Definition 2.3.2. *If $x \in X$ is periodic, then $\Lambda^+(x) = \Lambda^-(x) = \gamma(x)$.*

Example 2.3.3. *Examples of Limit Sets.*

Consider the differential system defined in $\mathbb{R}^2$ by the differential equations in Polar coordinates:

$$\frac{dr}{dt} = r(1-r), \frac{d\theta}{dt} = 1 \tag{2.14}$$

It can easily be verified that the solutions are unique and all solutions are defined on $\mathbb{R}$. Thus 2.14 defines a dynamical system. The trajectories are shown in Figure 3.1. These consist of: (i) a critical point, namely the origin 0, (ii) a periodic trajectory γ coinciding with the unit circle, (iii) spiraling trajectories through each point $P = (r, \theta)$ with $r \neq 0, r \neq 1$. For points P with $0 < r < 1$, $\Lambda^+(P)$ is the unit circle and $\Lambda^-(P)$ is the singleton $\{0\}$. For points P with $r > 1, \Lambda^+(P)$ is the unit circle and $\Lambda^-(P) = \emptyset$.

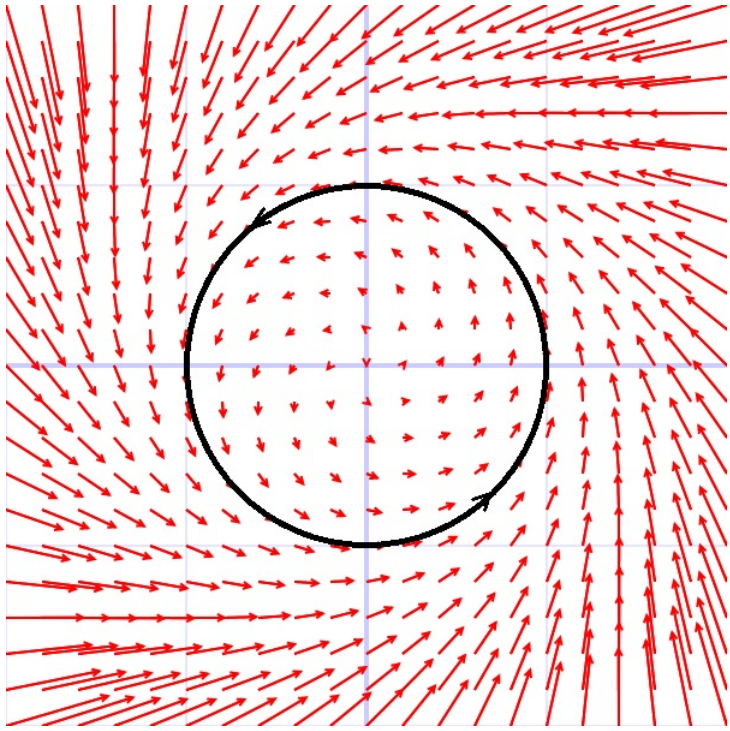

Figure 2.1:

The differential system can be redefined in Cartesian coordinates:

$$\frac{dx}{dt} = \frac{d}{dt}(rcos\theta) = \frac{dr}{dt}cos\theta - rsin\theta\frac{d\theta}{dt} \tag{2.15}$$

$$\frac{dy}{dt} = \frac{d}{dt}(rsin\theta) = \frac{dr}{dt}sin\theta + rcos\theta\frac{d\theta}{dt} \tag{2.16}$$

$$\frac{dx}{dt} = r(1-r)cos\theta - rsin\theta = r(1-r)cos\theta - rsin\theta \tag{2.17}$$

$$\frac{dy}{dt} = r(1-r)sin\theta + rcos\theta = r(1-r)sin\theta + rcos\theta \tag{2.18}$$

$$\frac{dx}{dt} = x - x\sqrt{x^2+y^2} - y \tag{2.19}$$

$$\frac{dy}{dt} = y - y\sqrt{x^2+y^2} + x \tag{2.20}$$

$$\frac{dy}{dx} = \frac{y - y\sqrt{x^2+y^2} + x}{x - x\sqrt{x^2+y^2} - y} \tag{2.21}$$

Chapter 3

Stability Theory

3.1 Stability and Attraction for Compact Sets

Definition 3.1.1. *With a given M we associate the sets*

$$A_\omega(M) = \{x \in X : \Lambda^+(x) \cap M \neq \emptyset\}, \tag{3.1}$$

$$A(M) = \{x \in X : \Lambda^+(x) \neq \emptyset \text{ and } \Lambda^+(x) \subset M\}, \tag{3.2}$$

$$A_u(M) = \{x \in X : J^+(x) \neq \emptyset \text{ and } J^+(x) \subset M\}, \tag{3.3}$$

$$J^+(x) = \{y \in X : \exists\{x_n\} \in X \text{ and } \{t_n\} \in R^+ \text{ s.t. } x_n \to x,\ t_n \to +\infty, \text{ and } x_n t_n \to y\} \tag{3.4}$$

The sets $A_\omega(M), A(M)$, and $A_u(M)$ are respectively called the ***region of weak attraction***, ***attraction***, *and* ***uniform attraction*** *of the set M. Moreover, any point x in $A_\omega(M), A(M)$, or $A_u(M)$ may respectively be said to be* ***weakly attracted***, ***attracted***, *and* ***uniformly attracted*** *to M.*

Proposition 3.1.2. *Given M, a point x is weakly attracted to M if and only if there is a sequence $\{t_n\} \in \mathbb{R}$ with*

$$t_n \to +\infty \text{ and } \varrho(t_n.x, M) \to 0. \tag{3.5}$$

A point x is attracted to M if and only if

$$\varrho(t.x, M) \to 0 \text{ as } t \to +\infty, \tag{3.6}$$

A point x is uniformly attracted to M if and only if for every neighborhood V of M there is a neighborhood U of x and a $T > 0$ with

$$Ut \subset V \text{ for } t \geqq T. \tag{3.7}$$

Theorem 3.1.3. *For any given M,*

$$A_\omega(M) \supset A(M) \supset A_u(M), \tag{3.8}$$

$$\text{the sets } A_\omega(M),\ A(M),\ \text{and } A_u(M) \tag{3.9}$$

are invariant.

Definition 3.1.4. *A given set M is said to be*

$$\text{a } \textbf{\textit{weak attractor}} \text{ if } A_\omega(M) \text{ is a neighborhood of } M, \tag{3.10}$$

$$\text{an } \textbf{\textit{attractor}} \text{ if } A(M) \text{ is a neighborhood of } M, \tag{3.11}$$

$$\text{a } \textbf{\textit{uniform attractor}} \text{ if } A_u(M) \text{ is a neighborhood of } M, \tag{3.12}$$

$$\textbf{\textit{stable}} \text{ if every neighborhood } U \text{ of } M \text{ has a positively invariant neighborhood } V \text{ of } M, \tag{3.13}$$

$$\textbf{\textit{asymptotically stable}} \text{ if it is stable and is an attractor,} \tag{3.14}$$

$$\textbf{\textit{unstable}}, \text{ if it is not stable.} \tag{3.15}$$

3.2 Lyapunov Functions: Characterization of Asymptotic Stability

The basic feature of the stability theory of a Lyapunov function is that one seeks to characterize stability and asymptotic stability of a given set in terms of a non-negative scalar function defined on a neighborhood of the given set and decreasing along its trajectories. It is in general not possible to characterize stability and the various attractor properties by means of continuous functions. However, in the case of asymptotic stability one can give very strong theorems.

Theorem 3.2.1. *A compact set $M \subset X$ is asymptotically stable if and only if there exists a continuous real-valued* ***Lyapunov function*** *Φ defined on a neighborhood N of M such that*

$$\Phi(x) = 0 \text{ if } x \in M \text{ and } \Phi(x) > 0 \text{ if } x \notin M \tag{3.16}$$

$$\Phi(t.x) < \Phi(x) \text{ for } x \notin M,\ t > 0 \text{ and } [0,t].x \subset N. \tag{3.17}$$

How does one construct a Lyapunov function? The following is a list of some standard ways to create such a function:

(1) If the dynamical system is a model of a physical system, try computing the energy at each state of the system.

(2) If the state vector is x and the fixed point is 0, let $V(x) = x_1^2 + x_2^2 + \cdots + x_n^2$; this is the square of the distance from x to 0.

(3) If the fixed point isn't 0 but rather $\tilde{x}$, the same idea still applies.

Let:

$$V(x) = (x_1 - \tilde{x_1})^2 + (x_2 - \tilde{x_2})^2 + \cdots + (x_n - \tilde{x_n})^2 \tag{3.18}$$

This squared distance idea sometimes won't work, so try more complicated ideas. For example, try

$$V(x) = a_1(x_1 - \tilde{x_1})^2 + a_2(x_2 - \tilde{x_2})^2 + \cdots + a_n(x_n - \tilde{x_n})^2, \tag{3.19}$$

where $a_1, a_2, \cdots, a_n$ are positive numbers. To determine what these numbers are, work backwards from $\dfrac{dV}{dt} < 0$. If that doesn't work, try

$$V(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x_i - \tilde{x_i})(x_j - \tilde{x_j}), \tag{3.20}$$

where the a_{ij}'s have the following property: The matrix $A = [a_{ij}]$ is symmetric and has positive eigenvalues. The positive eigenvalues ensure that $V(x) > 0$ $\forall x \neq \tilde{x}$.

Suppose that a given system has the form $x' = f(x)$ with fixed point $\tilde{x}$. The goal is to find a function $h(x)$ for which:

$$f(x) = - \bigtriangledown h(x), \tag{3.21}$$

$$h(\tilde{x}) = 0, \tag{3.22}$$

$$h(x) > 0 \ \forall \ x \neq \tilde{x} \tag{3.23}$$

If such a function exists, then it is a Lyapunov function and $\tilde{x}$ is stable. In order for 3.21 to hold, $\dfrac{\partial f_i}{\partial x_j} = \dfrac{\partial f_j}{\partial x_i}$. If this is the case, use integration to try to recover the function h. Adjust arbitrary constants in the formula to make condition 3.22 true, then check to see if 3.23 holds.

Refer back to 2.14. There is a very simple Lyapunov function for this system, given by the distance from any point to the unit circle:
Let $M = \{(x, y) : x^2 + y^2 = 1\}$ and define, for any point,

$$(x, y) \in \mathbb{R}^2, \Phi(x, y) = min\{\sqrt{(x-a)^2 + (y-b)^2} : (a, b) \in M\}. \tag{3.24}$$

Clearly, this function is continuous and decreasing along all trajectories.

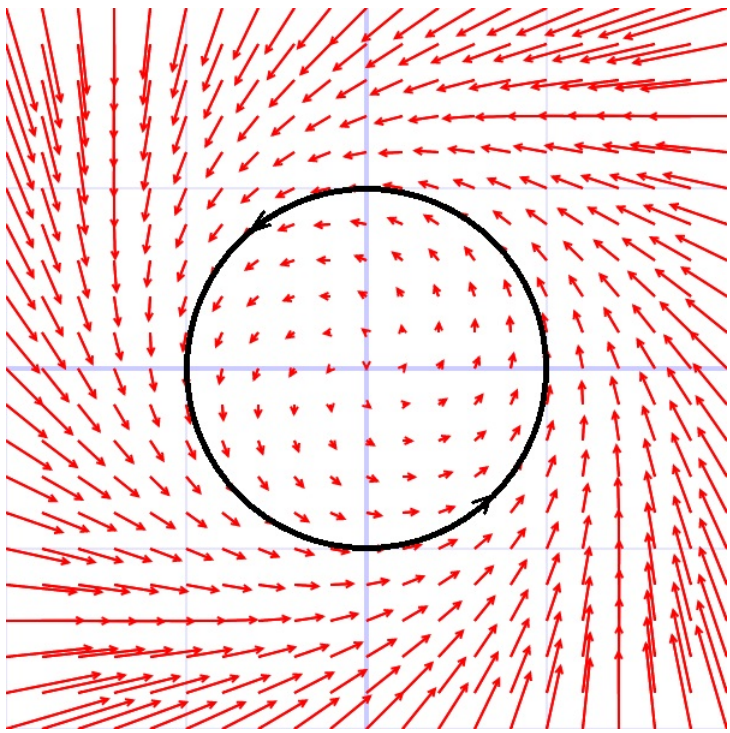

Figure 3.1:

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