Deepening Understanding of Quadratics Through Bruner's Theory of Representation

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Abstract

Bruner's Theory of Representation is typically applied in early childhood education, but it can also be beneficial in secondary education as well. In Bruner's Theory learners go from a tangible, action-oriented stage of learning to a symbolic and abstract stage of learning. By using this theory, learners can build new knowledge upon knowledge they've previously learned. This can lead to a better understanding of what students are learning. A way to apply this in an upper-level math class is through manipulatives. The purpose of my study is to show how successful Bruner's Theory is in increasing understanding and creating connections in mathematics when applied to secondary and upper-level math classes. Students from the Special Education Cohort at Georgia College were given a lesson on completing the square using Algebra Tiles and given a pre- and post- assessment to assess their learning and understanding. Through this project we will see the impact of applying Bruner's Theory in the classroom.

Deepening Understanding of Quadratics Through Bruner's Theory of Representation

Very often teachers in upper-level math classes teach math by giving formulas and telling students to memorize a procedure. Students are leaning procedure rather than getting a complete understanding of the topics. Through Bruner's Theory of Representation students are led through steps to gain their whole understanding. Manipulatives are very often used in elementary classes but rarely used in upper-level math classes, even though there are benefits to using them. Manipulatives are a great way to lead through the steps of understanding. Through this study I want to discover how can teachers apply Bruner's Theory in their classroom, especially upper-level mathematics classes? Does applying the first two levels of Bruner's Theory instead of going straight to an abstract, symbolic way of thinking increase understanding? Does applying Bruner's Theory help create connections in mathematic understanding?

Literature Review

Gningue et al use Bruner's Theory of Representation to teach pre-algebra and algebra concepts. This theory explains that, when faced with new material, a child goes through three stages of representation and follow the progression from an enactive to an iconic to a symbolic representation. In the enactive stage, the child needs action with materials in order to understand a concept. The iconic level a child creates mental representations of the objects but doesn't manipulate them. Finally, in the symbolic level the child strictly manipulates symbols and does not need to manipulate the objects (Gningue et al, 2014). While this theory is used in children's development and learning, it can be applied to when students are learning new material. The article, "Research on the Benefits of Manipulatives" explains three stages when using mathematical concepts that align with Bruner's theory. There is the concrete stage, in which students are introduced to a manipulative and they explore a concept using the manipulatives.

Then in the representational stage, the mathematical concept is representing using pictures to stand for the manipulative and students should demonstrate how they can both visualize and communicate the concept at a pictorial level. Finally, in the abstract level, symbols (numerals, operation signs, etc.) are used to express the concept in symbolic language ("Research on the Benefits of Manipulatives", 2017). Manipulatives are able to give students a tangible experience, so they are able to develop abstract reasoning. Manipulatives allow students to hone their mathematical skills and to connect mathematical ideals. Bruner's Theory is built upon the Constructivist Theory. Learning is an active process where students construct new ideas by building upon their past knowledge (Culatta, 2018). The Constructivist Theory relates to schemas, which are cognitive structures that help organize the world. When learning, people build on their schemas of that subject and make new connections with what they have learned.

It is important for students to have sense making and to create "well-connected schemas" in mathematics. This is done by not jumping straight to a formula but through a process where one can develop ideas from a concrete example to an abstract and symbolic idea. In a study by Palatnik and Koichu, students looked at a certain algebraic thinking problem about cutting up a pizza. They were to find the largest number of pieces that can be obtained by n straight cuts. They were first given a specific value of n. They went through this example where they drew pictures and tables. They went through steps that allowed them to find patterns and generalize and come up with a formula for this problem. When they had a discussion later about their findings, they were happy and excited to explain why it works. One student explained that if you only have a formula and you do not understand it or know it's meaning it is not interesting (Palatnik & Koichu, 2017). In another study on schematic-theoretic view of problem solving in math, Steele and Johanning looked at students with what they called well-connected schemas and partially formed schemas. When going through a series of problems that on the surface do not seem to have a lot to do with each other, students with well-connected schemas were able to apply what they did in the last problem to the next to help them. When given a few concrete examples on the first problem that wanted to know how many squares there were in a border, they were eventually able to make connections of what they are doing and generalize it into a formula. In the next problem students were able to take the formula generated in last problem and apply it to the next and so on for the next problems. Without their knowledge of how they got the formula or why it worked they would not have been able to connect it to the next problem. Those with partially formed schemas, did not make those connections in the beginning so even if they had a formula it was hard for them to connect to the next problems what they just did (Steele & Johanning, 2004).

Method

Participants

The participants for the taken from Dr. Samples' Math education course for the Special Education pre-service teachers at Georgia College. Twenty-one students were in person and one joined online through Zoom during the lesson. Most of the class were considered juniors in college but there were also two seniors. This project was intended to be done in a high school algebra class but due to the current situation of COVID-19, it was much more possible to get into a class at Georgia College.

Procedure

I started the lesson by handing out a preassessment that tested their knowledge on solving quadratic equations (see Appendix A). The preassessment asked students what is a quadratic, to solve quadratic equations, and to find the equation that matched the graph given. After the preassessment I started my lesson with defining what is a quadratic. Throughout the lesson I continued to use Bruner's Theory even during small discussions. For example, when defining a quadratic, I started by asking for some examples of a quadratic and then moved on to creating a definition of a quadratic. I reviewed some key terms of quadratics and ways to solve a quadratic equation. Since the focus of the lesson was on completing the square, we also discussed finding the area of a square.

We moved on to an online manipulative program on BrainingCamp.com and the lesson worksheet (see Appendix B). The students have been using this program since the beginning of the semester, so I did not have to teach them how to use it. I did explain the three rectangles we would be using and their dimensions, connecting it back to the area. Using the algebra tiles, I first gave the students the equation $x^2 + 4x = 0$. I wanted the students to use the algebra tiles to represent this equation by making it as close to a square as possible (see Figure 1).

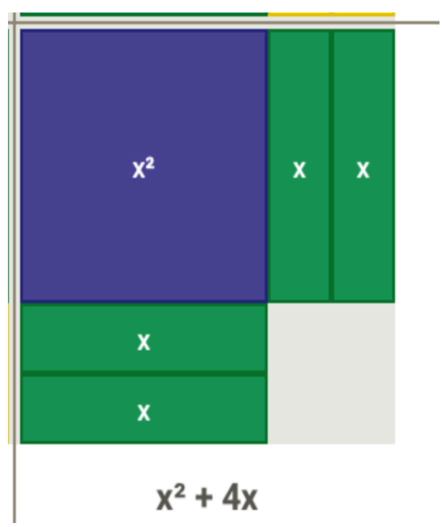

Figure 1. Representing $x^2 + 4x = 0$ with algebra tiles.

From there I asked the students what it would take to complete the square. We then discussed what we did with the algebra tiles so that we could connect it to how we algebraically solve the equation (see Figure 2 and 3).

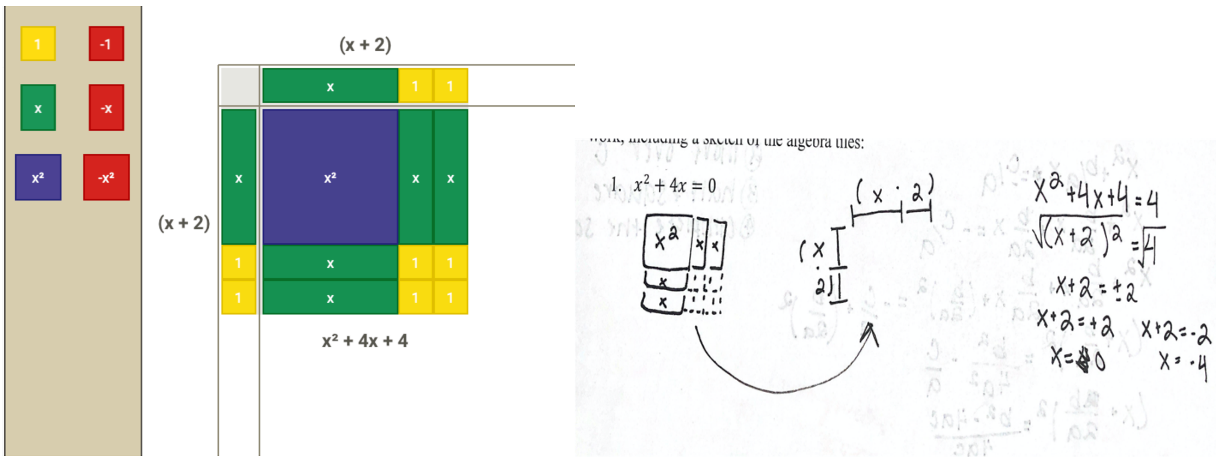

Figure 2 and 3. Figure 2 shows the completed square with the lengths of the sides and the total are found. Figure 3 shows the algebraic work representing what we did with the algebra tiles.

I had the students do two more examples, $x^2 + 6x = 0$ and $4x^2 + 24x + 20 = 0$. The students were able to work with the people around them and I walked around listening to discussion and answering any questions the students had. I had one student come up and explain their answer and how they got it. For the equation, $4x^2 + 24x + 20 = 0$, I asked students to make this equation look as much like the last equations we did. By dividing the equation by 6 and subtracting 5 from both sides of the equation students were able to come up with, $x^2 + 6x = -5$. With the added constant students had to complete the square. Students found two different methods to take while using the algebra tiles. One way was by subtracting the five to the other side like we came up with as a class together (see Figure 4). This way we are able to complete the square the same way we did in the previous problem but just adding nine and negative five on the right side of the equation (see Figure 5).

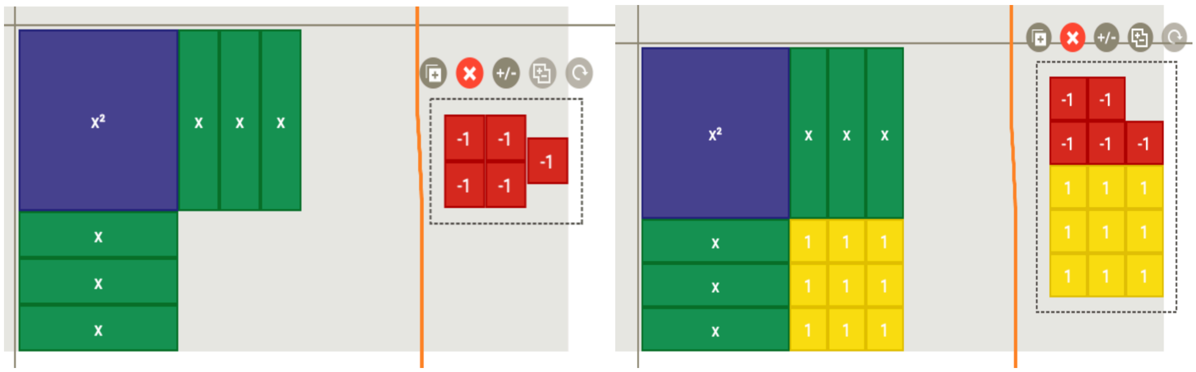

Figure 4 and 5. Figure 4 shows $x^2 + 6x = -5$ with algebra tiles. The red line represents an equal sign. Figure 5 shows the square completed by adding 9 to both sides of the equation (Harris & Brown, 2011).

The other method used was leaving the 5 on the left side of the equation and completing the square that way (see Figure 6). We can still complete the square, but it is a smaller area needed to be completed (see Figure 7).

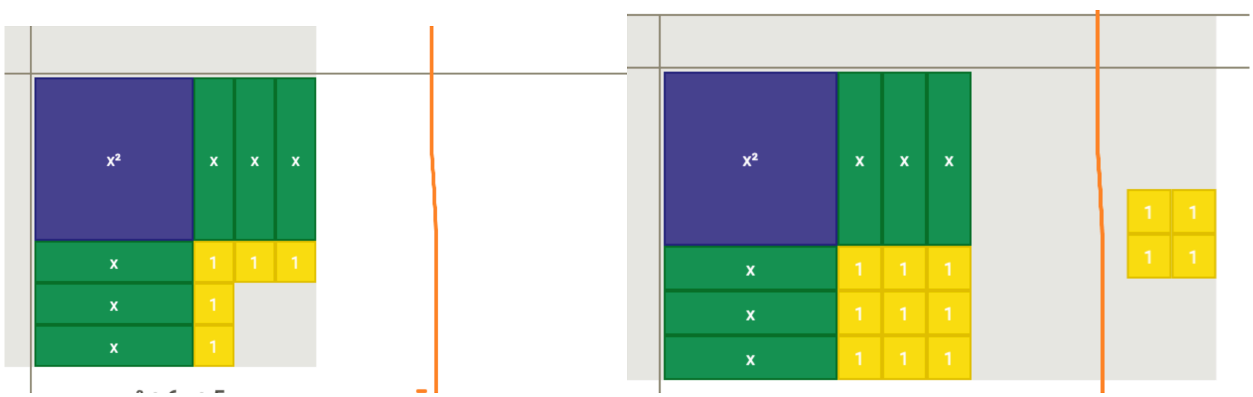

Figure 6 and 7. Figure 6 shows $x^2 + 6x + 5 = 0$ with algebra tiles and Figure 7 shows this completed by adding 4 blocks to complete the square.

We then showed what we did with the algebra tiles algebraically (see Figure 8).

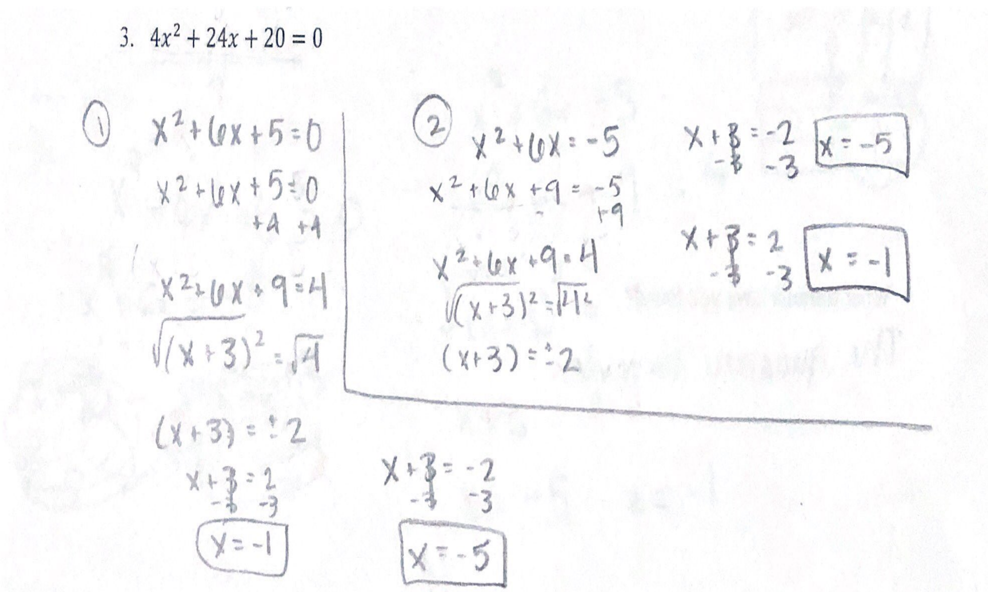

Figure 8. Algebraic work of the third example. 1 shows the work if you left the 5 on the left side of the equation and 2 shows the work if you subtracted the 5 to the right side. Either way you get the same answer.

From the examples, we moved to generalizing completing the square. We still used the algebra tiles to represent what we were doing but had to realize that you could not count the algebra tiles. Instead the algebra tiles had to represent arbitrary numbers. From completing the square, the equation converts to the vertex form of a quadratic equation. From the vertex form, we solved for x to get the quadratic formula, which can be used to solve any quadratic equation for x.

After the lesson, the time ran out, so I was unable to give the post-assessment in class, so the students turned in the post-assessment the next time they met for class. The post-assessment has the same questions as the pre assessment but also included asking if they have seen what we did in the lesson (see Appendix A).

Results

The students pre- and post- assessments were graded with a rubric with a scale from zero to five points for a total of 35 possible points (see Appendix C). Of the twenty-two students, 90.9% of them did better or got the same score on the post-assessment. The two students who did worse both had good scores on the pre-assessment and seemed to not really care about doing the post-assessment and seemed to rush through it. Overall, there was a 35.9% increase in scores with an average score of 15.7 (SD = 8.14) on the pre-assessment and an average score of 24.4 (SD = 6.2) on the post-assessment (see Figure 9).

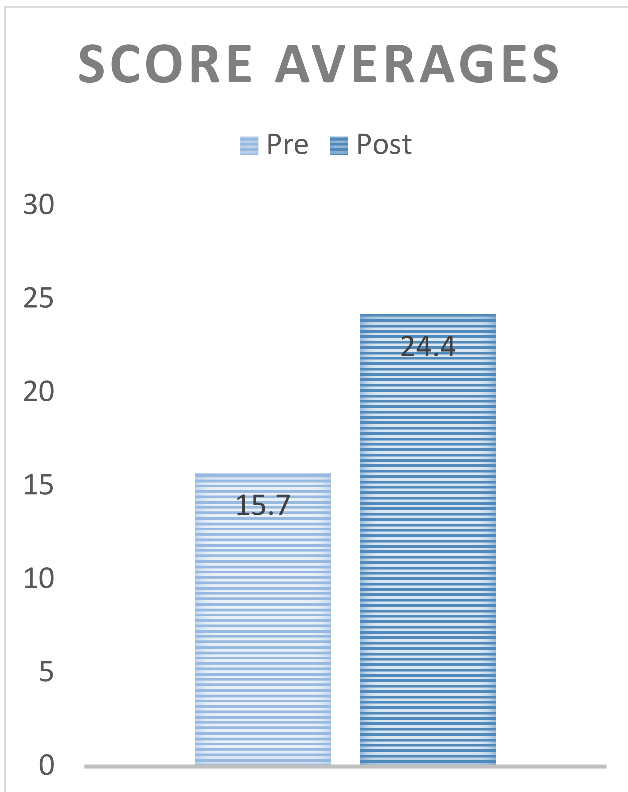

Figure 9. Pre vs Post assessment average scores

After grading the pre- and post-assessment and starting to organize the data, I realized I should have specifically asked the students to complete the square or use the lesson as I did not get to fully assess their gain in understanding of completing the square, but I was able to assess their understanding of quadratics as a whole. When grading the solving for x questions, I broke down each question in the method used to solve the equation. The methods that I found most used were factoring, quadratic formula, completing the square, no real method used, and no attempt made.

Question: $x^2 - 8x + 16 = 0$

There was a 19% increase in scores from the pre- to the post-assessment. There was an average of 2.52 (SD = 1.9) in the pre-assessment and an average of 3.00 (SD = 1.8) in the post-assessment. The following figure shows the percent of students using each method in the pre- versus the post- (see Figure 10).

	Factoring	**Quadratic Formula**	**Completing the Square**	**No Real Method**	**No attempt**
Pre	40.9%	4.5%	4.5%	36.4%	13.6%
Post	36.4%	13.6%	31.8%	13.6%	4.5%

Figure 10.

Question: $x^2 + 5x + 4 = 0$

There was a 51.7% increase in scores from the pre- to the post-assessment. There was an average of 2.38 (SD = 1.9) in the pre-assessment and an average of 3.61 (SD = 1.9) in the post-assessment. The following figure shows the percent of students using each method in the pre versus the post (see Figure 11).

	Factoring	Quadratic Formula	Completing the Square	No Real Method	No attempt
Pre	31.8%	4.5%	4.5%	45.5%	13.6%
Post	59.1%	13.6%	13.6%	0%	13.6%

Figure 11.

Question: $3x^2 - 75 = 0$

There was a 39.5% increase in scores from the pre- to the post-assessment. There was an average of 2.76 (SD = 1.8) in the pre-assessment and an average of 3.85 (SD = 1.8) in the post-assessment. The following figure shows the percent of students using each method in the pre versus the post (see Figure 12). The method of using inverse was included in the factoring category as many of the students solved this question by using the inverse of a square is the square root.

	Factoring/Inverse	Quadratic Formula	Completing the Square	No Real Method	No attempt
Pre	59.1%	4.5%	0%	18.2%	18.2%
Post	68.2%	13.6%	13.6%	0%	13.6%

Figure 12.

Question: $x^2 + 8x + 15 = 0$

There was a 60% increase in scores from the pre- to the post-assessment. There was an average of 2.23 (SD = 2.0) in the pre-assessment and an average of 3.57 (SD = 1.8) in the post-assessment. The following figure shows the percent of students using each method in the pre versus the post (see Figure 13).

	Factoring	**Quadratic Formula**	**Completing the Square**	**No Real Method**	**No attempt**
Pre	40.9%	0%	4.5%	31.8%	22.7%
Post	54.5%	13.6%	18.2%	4.5%	9.1%

Figure 13.

Each question follows the same pattern of most of the students using the method of factoring in both the pre- and post- assessment. There was an increase from the pre to post assessment of the number of students using the quadratic formula and completing the square, which was what was taught in the lesson. There was also a decrease from the pre- to the post-assessment in the number of students who did not know what to do when solving the equation.

Defining a Quadratic Equation.

When asked to define what a quadratic equation is, there was a 64.8% increase in scores from the pre- to post-assessment. There was an average score of 1.76 (SD = 1.33) in the pre assessment and an average score of 2.90 (SD=1.30) in the post-assessment. In the pre-assessment many of the answers were vague and did not describe much of anything (see Figure 14). Only two students gave a definition that I gave a score of 4 or 5.

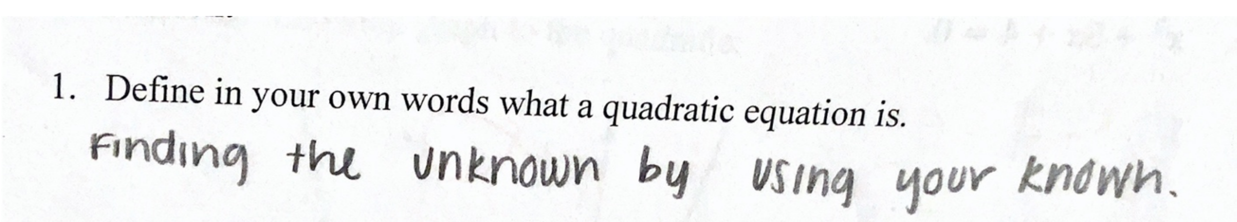
1. Define in your own words what a quadratic equation is.
Finding the unknown by using your known.

Figure 14. This is an example of one student's definition from the pre-assessment. There is not much explanation and is very vague.

In the post-assessment, I got mixed results. Many of the definitions were clear and gave good explanations of what is a quadratic (see Figure 15). Students gave definitions, examples, and graphs. There were also a few students that gave vague definitions.

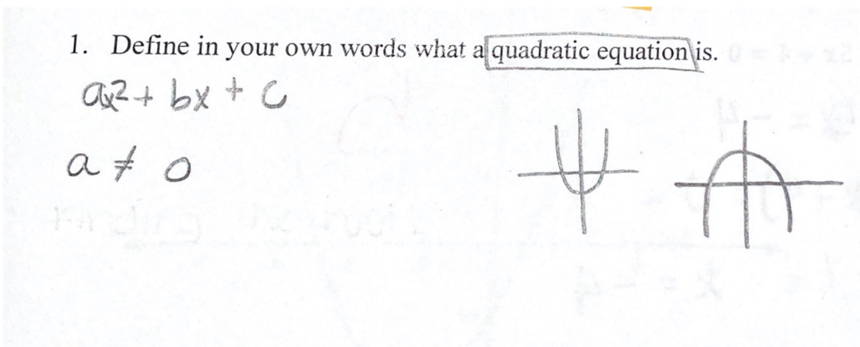

Figure 15. This is an example of one student's definition from the post-assessment that I considered a good definition. This student included graphs as well as the generalized equation.

Students Who Have Seen this Lesson Before

Of the 22 students, four of the students have seen the lesson we did with completing the square. In the pre-assessment the 4 students had an average score of 15.8 (SD = 4.2). Each student had a score that I considered a moderate to good understanding in the pre-assessment. In the post-assessment the 4 students had an average score of 22.5 (SD = 6.6).

Implications and Conclusions

Even with a small sample, applying Bruner's Theory shows positive effects in a classroom. However, this theory does need to be established and developed within a classroom. Students who have only been learning by being given a formula and told to memorize it and do procedure are not as used to developing and building ideas on top of each other. This class has been using manipulatives and developing mathematical ideas into abstract ways of thinking throughout this semester, so I had an advantage with teaching this lesson. Manipulatives are a great way to apply the enactive, concrete stage. They are engaging, fun, and a great way to make a lesson memorable. From looking at the percentage of students using certain methods, there was an increase in students who knew what to do. After the lesson, students started making connections back to what they learned before, and their schema for quadratics was better connected. This lesson did help students when it came to solving quadratics. I cannot speak to their understanding of completing the square since I did not specifically test this. If I were to repeat this research, I would change the pre and post-assessment questions to include specifically asking to complete the square. I would also want to use Bruner's Theory while teaching the whole unit on quadratics. There would be more clear evidence of how this theory works in students understanding, especially because this theory is about building upon past knowledge so starting from the beginning of the unit would be beneficial. It may be interesting to look into the long-term benefits of applying Bruner's Theory. Even though it was a small sample of students who said they have seen this lesson before, they all had moderate to good understanding of quadratics on the pre-assessment. This could be because these students had a well-connected schema of quadratics and completing the square due to this lesson being taught to them before.

Applying Bruner's Theory in your class is beneficial as long as it is an established method in your class. Students are able to see where these formulas come that are used in math, like the quadratic formula. Like a student said in Palatnik and Koichu study, "'When we have a formula, but don't know its meaning, it is not interesting. If we knew how the formula is constructed, we would know it 100%'" (2017). Allowing for multiple ideas and ways to solve problems is important in the classroom and discussion. Like the example we did in class, leaving the five on the left side of the equation made more sense to the student who asked about it.

Teachers should allow for multiple ways to solve problems in their classroom. If it makes more sense to do it one to a student, why should we hinder a student's understanding? Discussion of multiple ways to solve a problem can help students who are confused and may not want to speak up. As long as a student can understand why the ways work there should be no issue in a student working a problem a different way. It is also important to make it clear what you want from students. I was unable to properly assess their understanding due to how I formatted the pre- and post-assessments. Going off that however, I taught this lesson to pre-service teachers. They saw the lesson and saw the value in it, but most chose not to use it in their work.

This theory is used for young learners and is not commonly used for upper level classes even though there is benefits to using Bruner's Theory. By applying this throughout all math classes from elementary school through upper level learning, students would gain better understanding and appreciation for math. Students would get to see where these formulas come from in math and why they work. They will not be just memorizing a formula and procedurally going through math without much comprehension of what they are actually doing.

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Appendix A

Pre-and Post-Assessment

1. Define in your own words what a quadratic equation is.

2. What are you finding or what does it mean when you solve a quadratic?

3. What is you preferred method for solving a quadratic equation?

4. Solve the following equations for x:

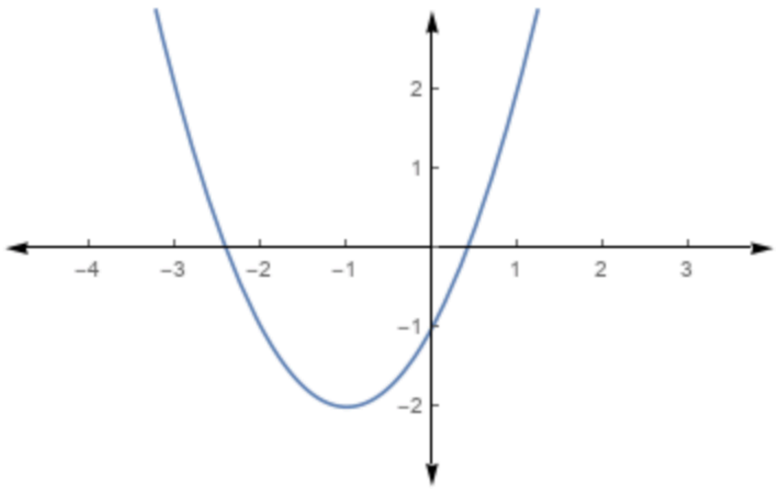
    a. $x^2 - 8x + 16 = 0$

    b. $x^2 + 5x + 4 = 0$

    c. $3x^2 - 75 = 0$

    d. $x^2 + 8x + 15 = 0$

5. Match the following graph to the quadratic:



a. $(x+2)^2 - 1$

b. $x^2 - 2x + 1$

c. $(x+1)^2 - 2$

d. $x^2 + 4x + 3$

6. How have you learned the quadratic equation in the past? Have you seen what we learned today? (Post-assessment only)

Appendix B

Lesson Worksheet

Exploration: Using Algebra Tiles to find the Quadratic Equation

Use algebra tiles to complete the square and solve for x for the following expression show all work, including a sketch of the algebra tiles:

1. $x^2 + 4x = 0$

2. $x^2 + 6x = 0$

3. $4x^2 + 24x + 20 = 0$

4. $ax^2 + bx + c = 0$

Find the vertex form of $ax^2 + bx + c$ and then use it to solve the equation for x.

What formula have you found?

Appendix C

Rubric

	0	1	2	3	4	5
Evaluate	No attempt made to answer the question	No mathematical logic in the attempt to solve the equation; no work shown	Work shows some understanding of problem, method or work unclear, answer may not be correct	Proper method used but many mistakes, shows some understanding	Proper method to solve quadratic and shows understanding of problem, some mistakes made	No errors; correct use of methods to solve quadratic and shows understanding or problem
Explain	No attempt made to answer the question	Only used words to explain. Explanation has no mathematical logic	Used words to explain but there is some understanding, but explanation is unclear or wrong	Used, just words pics numbers or graphs shows understanding many errors	Used words, pictures, numbers, and/or graphs to explain and shows understanding, some errors	No errors; explanation used words, pictures, numbers and/or graphs to show understanding