Understanding Fractional Integrals and their Applications

An Exploration of Fractional-Order Integrals

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1 Abstract

Fractional-Order Integrals have been considered since the beginning of Calculus, but have not been truly worked on until recently. This field has gained traction as some physical systems have been found to be indescribable using traditional methods. By defining and utilizing Riemann-Louisville fractional integrals and derivatives, Caputo fractional derivatives, and several others, the process of solving these physical systems is made possible. In this paper, the focus will be on defining Newton's Second Law using fractional integrals and derivatives and utilizing this new definition to solve the fractional harmonic oscillator.

2 Introduction of Riemann Integrals

First it is important understand the Riemann-Louisville operator of fractional integration. In order to define this operation, consider the following:

Start with a normal integral

$$\int_a^x f(t)dt$$

Now integrate again

$$\int_a^x \int_a^v f(t)dtdv$$

By changing the order of integration

$$\int_a^x \int_t^x f(t)dvdt = \int_a^x f(t)(x-t)dt$$

Integrating a third time starts to yield a noticeable pattern

$$\begin{aligned}\int_a^x \int_a^w f(t)(w-t)dtdw &= \int_a^x \int_t^x f(t)(w-t)dwdt \\ &= \int_a^x f(t)\frac{(x-t)^2}{2}dt\end{aligned}$$

The pattern can be written as:

$$J_a^n f(x) = \frac{1}{(n-1)!}\int_a^x f(t)(x-t)^{n-1}dt,$$

for $n > 0$ and $J^0 f(x) = f(x)$.

For now it will suffice to take $a = 0$

$$J_0^n f(x) = \frac{1}{(n-1)!}\int_0^x f(t)(x-t)^{n-1}dt$$

2.1 Fractional Integral

Note that J_0^1 is defined such that

$$J^1 f(x) = \int_0^x f(t)dt$$

Thus the definition can be extended for the k^{th} integral such that:

$$J^k f(x) = \frac{1}{(k-1)!}\int_0^x (x-t)^{k-1} f(t)dt$$

Extending the integral for the $k+1$ yields

$$J^{k+1} = J^1 J^k = \int_0^x [\frac{1}{(k-1)!}\int_0^x (x-t)^{k-1} f(t)dt]dx.$$

By interchanging the order of integration, the resulting form is

$$J^{k+1} f(x) = \frac{1}{k!}\int_0^x (x-t)^k f(t)dt.$$

Since the function is true for 1 and $k+1$, by induction it holds for all n. Thus, the integral of $f(t)$ of order n is

$$J_0^n f(x) = \frac{1}{(n-1)!} \int_0^x f(t)(x-t)^{n-1} dt.$$

This can be extended for $v > 0$ by utilizing the Gamma function.

$$J^v f(t) = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t) dt, \tag{1}$$

where the Gamma Function is defined as $\Gamma(v) = \int_0^\infty x^{v-1} e^{-x} dx$. We define Riemann-Louisville fractional integral as (1).

2.2 Fractional Derivative

With the fractional integral defined it follows to define a fractional derivative. This can be done by combining the standard derivative with the fractional integral between 0 and 1. However, the order in which the operators are applied makes a difference. Applying the integral first yields

$$D^\alpha f(x) = D^n J^{n-\alpha} f(x)$$

which is know as the Riemann-Louiville derivative fractional derivative.

Applying the derivative first yields

$$D_*^\alpha f(x) = J^{n-\alpha} D^n f(x)$$

where n is the nearest integer greater than α. This is known as the Caputo derivative.

2.3 The Mittag-Leffler Function

The Mittag-Leffler Function is a special function that is dependent on two complex parameters α, β. When α is strictly positive, the function can be written as such

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}.$$

If α and β are both positive and real, then the series converges for all values of z. This is a very important special function in the field of fractional calculus, and will be used continuously throughout the entirety of this paper.

Another important property of the Mittag-Leffler function which will be used later in the paper is the Laplace transform defined as

$$\mathbb{L}(t^{\gamma-1} E_{\beta,\gamma}^{\delta}(\omega t^\beta); s) = s^{-\gamma}(1 - \omega s^{-\beta})^{-\delta},$$

where $E_{\beta,\gamma}^{\delta}(t)$ is the Generalized Mittag-Leffler (GML) function is defined as

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{j=0}^{\infty} \frac{(\gamma)_j z^j}{j!\Gamma(\alpha j + \beta)}, \quad \alpha, \beta \in \mathbb{C}, Re(\alpha), Re(\beta), Re(\gamma) > 0, \tag{2}$$

where $(\gamma)_j = \gamma(\gamma+1)\cdots(\gamma+j-1)$ (for $j = 0, 1, \ldots,$ and $\gamma \neq 0$) is the Pochammer symbol and $(\gamma)_0 = 1$ and

$$\mathbb{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)dt$$

is the Laplace transform. The Laplace transform of Caputo derivative follows the same form as a regular derivative.

3 Properties

Let $\alpha > 0, A, B \in \mathbb{R}$ and let f and g be functions whose fractional derivatives and integrals exist. Then they are linear such that:

$$J^\alpha(Af(t) + Bg(t)) = AJ^\alpha f(t) + BJ^\alpha g(t)$$

$$D^\alpha(Af(t) + Bg(t)) = AD^\alpha f(t) + BD^\alpha g(t)$$

$$D_*^\alpha(Af(t) + Bg(t)) = AD_*^\alpha f(t) + BD_*^\alpha g(t)$$

It is also important to note that there is a significant difference in the Caputo and Reimann-Louiville Derivatives. Riemann-Louville derivative of a constant is not zero, but the Caputo Derivative is zero since derivative is applied first.

4 Examples and Special Cases

In most cases, the fractional integral of a function is not easily expressed using familiar and understood functions. However, represented here are some of the functions that do have solutions that can be expressed using well known functions.

$$J^v t^n = \frac{\Gamma(n+1)}{\Gamma(n+v+1)} t^{n+v}$$

for $n > -1, v > 0$. The fractional integral of a constant follows from this formula.

$$\begin{aligned} J^v c &= c \lim_{n \to 0} \frac{\Gamma(n+1)}{\Gamma(n+v+1)} t^{n+v} \\ &= \frac{c(t^v)}{\Gamma(v+1)} \end{aligned}$$

For the fractional integral of an exponential, first expand e^t into its Taylor Series.

$$J^v e^t = J^v \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

Now bring the Fractional Integral into the summation to get

$$\sum_{n=0}^{\infty} \frac{1}{n!} J^v t^n$$

Now using the Fractional integral of a polynomial we get

$$\sum_{n=0}^{\infty} \frac{1}{n!} \frac{\Gamma(n+1)t^{n+v}}{\Gamma(n+v+1)}$$

Using the property of the Gamma functions and exponentials, and pulling out any constants out of the sum yields:

$$\begin{aligned} t^v \sum_{n=0}^{\infty} \frac{\Gamma(n+1)t^n}{\Gamma(n+1)\Gamma(n+v+1)} &= t^v \sum_{n=0}^{\infty} \frac{t^v}{\Gamma(n+v+1)} \\ &= t^v E_{1,v+1}(t) \end{aligned}$$

For the special case $v = 1$ w get

$$J^1 e^t = \int_0^t e^s ds = tE_{1,2}(t) = e^t - 1,$$

which agrees with regular integral.

For the fractional integral of $\sin(t)$, it is easiest to again use the Taylor Series Expansion such that

$$\begin{aligned} J^v \sin(x) &= J^v \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(2n+2)} J^v x^{2n+1} \end{aligned}$$

Let $m = 2n+1$ and apply the Fractional Integral to get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(m+1)} \frac{\Gamma(m+1)}{\Gamma(m+v+1)} t^{m+v} &= t^v \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(m+v+1)} t^m \\ &= t^v \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma((2n+1)+v+1)} t^{2n+1} \end{aligned}$$

Rearranging to a more usable form we have:

$$J^v \sin(x) = t^{v+1} \sum_{n=0}^{\infty} \frac{(-t^2)^n}{\Gamma(2n+v+2)} = t^{v+1} E_{2,v+2}(-t^2)$$

When $v = 1$ our result reduces to the regular integral:

$$J^1 \sin(t) = \int_0^t \sin(s) ds = t^2 E_{2,3}(-t^2) = 1 - \cos(t).$$

For the fractional integral of $\cos(t)$, it is easiest to once again use a Taylor Series Expansion:

$$J^v \cos(t) = J^v \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(2n+1)} J^v (t^2)^n$$

Letting $m = 2n$, and applying the Fractional Integral yields

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(m+1)} \frac{\Gamma(m+1)}{\Gamma(m+v+1)} t^{m+v} &= t^v \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(m+v+1)} t^m \\ &= t^v \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(2n+v+1)} t^{2n}\end{aligned}$$

Rearranging to a more usable form yields:

$$\begin{aligned}J^v \cos(t) &= t^v \sum_{n=0}^{\infty} \frac{(-t^2)^n}{\Gamma(2n+v+1)} \\ &= t^v E_{2,v+1}(-t^2)\end{aligned}$$

Again for the special case $v = 1$ our result agrees with regular integral:

$$J^1 \cos(t) = \int_0^t \cos(s) ds = \sin(t) = t E_{2,2}(-t^2).$$

5 Application

One particular application of fractional calculus is extending classical mechanics to a fractional case. First it is important to preface the work with the classical derivation of velocity using Newtons Second Law.

5.1 Classical Derivation

Start by rewriting in terms of an object of mass m, falling with an initial velocity v_0, and drag proportional to velocity kv.

$$F = ma = m\frac{dv}{dt} = mg - kv$$

Now divide by m and then use separation of variables to set up for a first order differential equation.

$$\frac{dv}{dt} = g - \frac{k}{m} v$$

Separating the variables we have

$$\frac{dv}{g - \frac{k}{m} v} = dt$$

Letting w be a placeholder variable for v, and setting up the integrals yields the form:

$$\int_{v_0}^{v} \frac{dw}{g - \frac{k}{m} w} = \int_0^t dt = t$$

Using u-substitution such that $u = g - \frac{k}{m} w$ and $dw = -\frac{m}{k} du$, rewriting the equation yields a form that is easily integrable

$$-\frac{m}{k} \int_{v_0}^{v} \frac{du}{u} = -\frac{m}{k} \ln(u)|_{v_0}^{v} = t$$

Replacing u and evaluating at v and v_0 yields

$$-\frac{m}{k}\ln(g-\frac{k}{m}w)|_{v_0}^{v}=-\frac{m}{k}\ln\left(\frac{g-\frac{k}{m}v}{g-\frac{k}{m}v_0}\right)=t$$

After exponentiating both sides, all that remains is algebra in order to solve for v.

$$\frac{g-\frac{k}{m}v}{g-\frac{k}{m}v_0}=e^{-\frac{kt}{m}} \implies -\frac{kv}{m}=-g+(g-\frac{kv_0}{m})e^{-\frac{kt}{m}}$$

Thus, the velocity of the object is given by

$$v=v(t)=\frac{mg}{k}-\left(\frac{mg}{k}-v_0\right)e^{\frac{-kt}{m}} \tag{3}$$

5.2 Fractional Derivation

It is important to note that the Caputo derivative of order α with $0<\alpha<1$ will be used moving forward. Starting with Newton's Second Law, it can be extended using fractional derivatives such that

$$F=mD_*^\alpha v$$

where F is a constant force, m is the mass of the body, and $D_*^\alpha v$ is the Caputo fractional derivative of the velocity. Now assume, for example, there is a body moving vertically in a medium with a resistance that is proportional to the fractional velocity. The equation of motion can be written as

$$F=mD_*^\alpha=mg-kv \tag{4}$$

where k is a constant for the ratio of resistance to fractional velocity. We denote the Laplace transform of a function f as

$$\mathbb{L}\{f(t)\}(s)=\int_0^\infty e^{-st}f(t)dt.$$

Start this problem by taking the Laplace transform of both sides.

$$\mathbb{L}(mD_*^\alpha)=\mathbb{L}(mg-kv)$$

Since the Laplace transform is a linear operator, constants can be pulled out

$$m[s^\alpha\mathbb{L}\{v\}-s^{\alpha-1}v_0]=\frac{mg}{s}-k\mathbb{L}\{v\}$$

Which implies

$$(ms^\alpha+k)\mathbb{L}\{v\}=\frac{mg}{s}+\frac{mv_0}{s^{1-\alpha}}$$

Solving for the Laplace transform and canceling out like terms yields

$$\begin{aligned}\mathbb{L}\{v\} &= \frac{mg}{s(ms^\alpha+k)}+\frac{mv_0}{s^{1-\alpha}(ms^\alpha+k)}\\ &= \frac{g}{s(s^\alpha+\frac{k}{m})}+\frac{v_0}{s^{1-\alpha}(s^\alpha+\frac{k}{m})}\end{aligned}$$

Now take the inverse Laplace, once again pulling out constants

$$v(t) = g\mathbb{L}^{-1}\left\{\frac{1}{s(s^{\alpha}+\frac{k}{m})}\right\} + v_0\mathbb{L}^{-1}\left\{\frac{1}{s^{1-\alpha}(s^{\alpha}+\frac{k}{m})}\right\}$$

Rewriting to a more useful form yields

$$v(t) = g\mathbb{L}^{-1}\left\{s^{-(1+\alpha)}(1+\frac{k}{m}s^{-\alpha})^{-1}\right\} + v_0\mathbb{L}^{-1}\left\{s^{-1}(1+\frac{k}{m}s^{-\alpha})^{-1}\right\}$$

Recall the Laplace transform of the Mittag-Leffler function

$$\mathbb{L}\{t^{\gamma-1}E^{\delta}_{\beta,\gamma}(\omega t^{\beta}); s\} = s^{-\gamma}(1-\omega s^{-\beta})^{-\delta}.$$

Note that these are the same form, thus it is possible to write the solution in terms of the Mittag-Leffler by deducing the terms γ, β, δ

$$v(t) = gt^{\alpha}E_{\alpha,\alpha+1}\left(\frac{-k}{m}t^{\alpha}\right) + v_0 E_{\alpha,1}\left(\frac{-k}{m}t^{\alpha}\right).$$

In the special case $\alpha = 1$, our above result reduces to

$$\begin{aligned}
v(t) &= gtE_{1,2}\left(\frac{-k}{m}t\right) + v_0E_{1,1}\left(\frac{-k}{m}t\right) \\
&= gt\sum_{n=0}^{\infty}\frac{(-kt/m)^n}{\Gamma(n+2)} + v_0\sum_{n=0}^{\infty}\frac{(-kt/m)^n}{\Gamma(n+1)} \\
&= \frac{mg}{k} - \frac{mg}{k}e^{\frac{-kt}{m}} + v_0 e^{\frac{-kt}{m}},
\end{aligned}$$

as given in (3).

5.3 Fractional Simple Harmonic Oscillator

Consider the Harmonic Oscillator problem where

$$F = mD_*^{2\alpha}x(t) = -m\omega^2 x(t)$$

This problem is solved similarly to the previous one. Start by taking the Laplace transform.

$$s^{2\alpha}\mathbb{L}\{x\} - s^{2\alpha-1}x(0) - s^{2\alpha-2}x'(0) = -\omega^2\mathbb{L}\{x\}$$

Rearranging we get

$$(s^{2\alpha}+\omega^2)\mathbb{L}\{x\} = s^{2\alpha-1}x(0) + s^{2\alpha-2}x'(0)$$

Solving for the Laplace transform yields

$$\mathbb{L}\{x\} = \frac{x_0}{s^{1-2\alpha}(s^{2\alpha}+\omega^2)} + \frac{x'(0)}{s^{2-2\alpha}(s^{2\alpha}+\omega^2)}$$

Now taking the inverse Laplace yields

$$x(t) = x(0)\mathbb{L}^{-1}\{s^{-1}(1+\omega^2 s^{-2\alpha})\} + x'(0)\mathbb{L}^{-1}\{s^{-2}(1+\omega^2 s^{-2\alpha})\}$$

Once again comparing to the Laplace of the Mittag-Leffler, they are of the same form, and thus can be rewritten as

$$x(t) = x(0)E_{2\alpha,1}(-\omega^2 t^{2\alpha}) + x'(0)tE_{2\alpha,2}(-\omega^2 t^{2\alpha}).$$

When $\alpha = 1$ our solution reduces to the solution of simple harmonic motion. This is the case since for $\alpha = 1$ we have

$$\begin{aligned}
x(t) &= x(0)E_{2,1}(-\omega^2 t^2) + x'(0)tE_{2,2}(-\omega^2 t^2) \\
&= x(0)E_{2,1}(-\omega^2 t^2) + \frac{x'(0)}{\omega}(\omega t)E_{2,2}(-\omega^2 t^2) \\
&= x(0)\cos(\omega t) + \frac{x'(0)}{\omega}\sin(\omega t).
\end{aligned}$$

6 References

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