Lie Algebras of Generalized Quaternion Groups

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Abstract

Every finite group has an associated Lie algebra. Its Lie algebra can be viewed as a subspace of the group algebra with certain bracket conditions imposed on the elements. If one calculates the character table for a finite group, the structure of its associated Lie algebra can be described. In this work, we consider the family of generalized quaternion groups and describe its associated Lie algebra structure completely.

1 Introduction

The Lie algebra of a group is a useful tool because it is a vector space where linear algebra is available. It is interesting to consider the Lie algebra structure associated to a specific group or family of groups. A Lie algebra is simple if its dimension is at least two and it only has $\{0\}$ and itself as ideals. Some examples of simple algebras are the classical Lie algebras: $\mathfrak{sl}(n)$, $\mathfrak{sp}(n)$ and $\mathfrak{o}(n)$ as well as the five exceptional finite dimensional simple Lie algebras. A direct sum of simple lie algebras is called a semi-simple Lie algebra. Therefore, it is also interesting to consider if the Lie algebra structure associated with a particular group is simple or semi-simple. In fact, the Lie algebra structure of a finite group is well known and given by a theorem of Cohen and Taylor [1]. In this theorem, they specifically describe the Lie algebra structure using character theory. That is, the associated Lie algebra structure of a finite group can be described if one calculates the character table for the finite group. In particular, the quaternions are a way to generalize complex numbers to higher dimensions. The quaternion group has many scientific applications; for example, aerospace and robotics engineers use them to model the positions and orientations of planes and robots. In this paper, we will begin with basic definitions and examples of Lie algebras of quaternions and then describe the Lie algebra structure of the generalized quaternion group completely.

2 Preliminaries

In order to describe the Lie algebra structure associated with the generalized quaternion group, we must first consider the definition of a Lie algebra.

Definition 2.1. Let L be a vector space over a field F. Then L is a **Lie algebra** if it possesses a bracket operation $[\ ,\ ] : L \times L \to L$ that satisfies the following axioms:

i. The bracket operation is bilinear.

ii. $[x, x] = 0$ for all $x \in L$.

iii. $[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = 0$ $(x, y, z \in L)$.

Note that (i.) and (ii.) imply anticommutativity, $[x, y] = -[y, x]$, and (iii.) is called the Jacobi identity.

With the definition of a Lie algebra, we can consider a few examples. The first is the set of all vectors in $\mathbb{R}^3$ under the bracket operation of the cross product. Let $x, y, z \in \mathbb{R}^3$ with $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, and $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$. The cross product is defined as the following $x \times y = \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix}$. Note that x and y are general vectors and $x \times y = \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix} \in \mathbb{R}^3$; therefore, the set of vectors in $\mathbb{R}^3$ is closed under the cross product. Next we will consider the three axioms.

i. First, let $a, b \in \mathbb{R}$. Then we have the calculation for bilinearity as follows,

$$[ax + by, z] = \begin{bmatrix} (ax_2 + by_2)z_3 - (ax_3 + by_3)z_2 \\ (ax_3 + by_3)z_1 - (ax_1 + by_1)z_3 \\ (ax_1 + by_1)z_2 - (ax_2 + by_2)z_1 \end{bmatrix} = \begin{bmatrix} ax_2z_3 + by_2z_3 - ax_3z_2 - by_3z_2 \\ ax_3z_1 + by_3z_1 - ax_1z_3 - by_1z_3 \\ ax_1z_2 + by_1z_2 - ax_2z_1 - by_2z_1 \end{bmatrix}$$

$$= \begin{bmatrix} a(x_2z_3 - x_3z_2) + b(y_2z_3 - y_3z_2) \\ a(x_3z_1 - x_1z_3) + b(y_3z_1 - y_1z_3) \\ a(x_1z_2 - x_2z_1) + b(y_1z_2 - y_2z_1) \end{bmatrix} = a[x, z] + b[y, z],$$

and $$[x, ay + bz] = \begin{bmatrix} x_2(ay_3 + bz_3) - x_3(ay_2 + bz_2) \\ x_3(ay_1 + bz_1) - x_1(ay_3 + bz_3) \\ x_1(ay_2 + bz_2) - x_2(ay_1 + bz_1) \end{bmatrix} = \begin{bmatrix} x_2ay_3 + x_2bz_3 - x_3ay_2 - x_3bz_2 \\ x_3ay_1 + x_3bz_1 - x_1ay_3 - x_1bz_3 \\ x_1ay_2 + x_1bz_2 - x_2ay_1 - x_2bz_1 \end{bmatrix}$$

$$= \begin{bmatrix} (ax_2y_3 - x_3y_2) + b(x_2z_3 - x_3z_2) \\ a(x_3y_1 - x_1y_3) + b(x_3z_1 - x_1z_3) \\ a(x_1y_2 - x_2y_1) + b(x_1z_2 - x_2z_1) \end{bmatrix} = a[x, y] + b[x, z].$$

Therefore, the cross product satisfies bilinearity.

ii. Next, consider an element bracketed with itself:

$$[x, x] = \begin{bmatrix} x_2x_3 - x_3x_2 \\ x_3x_1 - x_1x_3 \\ x_1x_2 - x_2x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So we have that the second axiom is satisfied.

iii. Consider the Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z[x, y]] = \left[x, \begin{bmatrix} y_2z_3 - y_3z_2 \\ y_3z_1 - y_1z_3 \\ y_1z_2 - y_2z_1 \end{bmatrix}\right] + \left[y, \begin{bmatrix} z_2x_3 - z_3x_2 \\ z_3x_1 - z_1x_3 \\ z_1x_2 - z_2x_1 \end{bmatrix}\right] + \left[z, \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix}\right]$$

$$= \begin{bmatrix} x_2(y_2z_3 - y_3z_2) - x_3(y_2z_3 - y_3z_2) \\ x_3(y_3z_1 - y_1z_3) - x_1(y_3z_1 - y_1z_3) \\ x_1(y_1z_2 - y_2z_1) - x_2(y_1z_2 - y_2z_1) \end{bmatrix} + \begin{bmatrix} y_2(z_2x_3 - z_3x_2) - y_3(z_2x_3 - z_3x_2) \\ y_3(z_3x_1 - z_1x_3) - y_1(z_3x_1 - z_1x_3) \\ y_1(z_1x_2 - z_2x_1) - y_2(z_1x_2 - z_2x_1) \end{bmatrix}$$

$$+ \begin{bmatrix} z_2(x_2y_3 - x_3y_2) - z_3(x_2y_3 - x_3y_2) \\ z_3(x_3y_1 - x_1y_3) - z_1(x_3y_1 - x_1y_3) \\ z_1(x_1y_2 - x_2y_1) - z_2(x_1y_2 - x_2y_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0. \end{bmatrix}$$

Consequently, the cross product satisfies the Jacobi identity.

Since we have shown that all of the properties for a Lie algebra are satisfied by the cross product, then we have that the vectors of $\mathbb{R}_3$ with the bracket operation of the cross product is indeed an example of a Lie algebra.

Next, consider the set of all 2×2 matrices of trace zero with entries in $\mathbb{C}$ with the bracket operation being $ab - ba$ is a special linear algebra known as $\mathfrak{sl}(2)$. Consider the matrices $a = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}$, $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}$, and $c = \begin{pmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{pmatrix}$, such that the entries are in $\mathbb{C}$. Note $a, b, c \in \mathfrak{sl}(2)$ and

$$\begin{aligned} ab - ba &= \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix} - \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix} \\ &= \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 - a_2b_3 \\ a_3b_1 - a_1b_3 & a_3b_2 + a_1b_1 \end{pmatrix} - \begin{pmatrix} b_1a_1 + b_2a_3 & b_1a_2 - b_2a_1 \\ b_3a_1 - b_1a_3 & b_3a_2 + b_1a_1 \end{pmatrix} \\ &= \begin{pmatrix} a_1b_1 + a_2b_3 - b_1a_1 - b_2a_3 & a_1b_2 - a_2b_3 - b_1a_2 + b_2a_1 \\ a_3b_1 - a_1b_3 - b_3a_1 + b_1a_3 & a_3b_2 + a_1b_1 - b_3a_2 - b_1a_1 \end{pmatrix} \in \mathfrak{sl}(2). \end{aligned}$$

Thus, we see that $\mathfrak{sl}(2)$ is closed under the bracket operation. Now consider the three axioms for a Lie algebra.

i. First, let $m, n \in \mathbb{C}$. The bilinearity calculation is as follows,

$$\begin{aligned} [ma + nb, c] &= \left[\begin{pmatrix} ma_1 + nb_1 & ma_2 + nb_2 \\ ma_3 + nb_3 & -ma_1 - nb_1 \end{pmatrix}, \begin{pmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{pmatrix}\right] \\ &= \begin{pmatrix} ma_1c_1 + nb_1c_1 + ma_2c_3 + nb_2c_3 & ma_1c_2 + nb_1c_2 - ma_2c_1 + nb_2c_1 \\ ma_3c_1 + nb_3c_1 + ma_1c_3 + nb_1c_3 & ma_3c_2 + nb_3c_2 + ma_1c_1 + nb_1c_1 \end{pmatrix} \\ &- \begin{pmatrix} c_1ma_1 + c_1nb_1 + c_2ma_3 + nc_2b_3 & c_1ma_2 + c_1nb_2 - c_2ma_1 - c_2nb_1 \\ c_3ma_1 + c_3nb_1 - c_1ma_3 + c_1nb_3 & c_3ma_2 + c_3nb_2 + c_1ma_1 + c_1nb_1 \end{pmatrix} \\ &= \begin{pmatrix} m(a_1c_1 + a_2c_3) + n(b_1c_1 + b_2c_3) & m(a_1c_2 - a_2c_1) + n(b_1c_2 + b_2c_1) \\ m(a_3c_1 + a_1c_3) + n(b_3c_1 + b_1c_3) & m(a_3c_2 + a_1c_1) + n(b_3c_2 + b_1c_1) \end{pmatrix} \\ &- \begin{pmatrix} m(c_1a_1 + c_2a_3) + n(c_1b_1 + c_2b_3) & m(c_1a_2 - c_2a_) + n(c_1b_21 - c_2b_1) \\ m(c_3a_1 - c_1ma_3) + n(c_3b_1 + c_1b_3) & m(c_3a_2 + c_1a_1) + n(c_3b_2 + c_1b_1) \end{pmatrix} \\ &= m[a, c] + n[b, c], \end{aligned}$$

and

$$\begin{aligned} [a, mb + nc] &= \left[\begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}, \begin{pmatrix} mb_1 + nc_1 & mb_2 + nc_2 \\ mb_3 + nc_3 & -mb_1 - nc_1 \end{pmatrix}\right] \\ &= \begin{pmatrix} a_1mb_1 + a_1nc_1 + a_2mb_3 + na_2c_3 & a_1mb_2 + a_1nc_2 - a_2mb_1 - a_2nc_1 \\ a_3ma_1 + a_3nc_1 - a_1mb_3 + a_1nc_3 & a_3mb_2 + a_3nc_2 + a_1mb_1 + a_1nc_1 \end{pmatrix} \\ &- \begin{pmatrix} mb_1a_1 + nc_1a_1 + mb_2a_3 + nc_2a_3 & mb_1a_2 + nc_1a_2 - mb_2a_1 + nc_2a_1 \\ mb_3a_1 + nc_3a_1 + mb_1a_3 + nc_1a_3 & mb_3a_2 + nc_3a_2 + mb_1a_1 + nc_1a_1 \end{pmatrix} \\ &= \begin{pmatrix} m(a_1b_1 + a_2b_3) + n(a_1c_1 + a_2c_3) & m(a_1b_2 - a_2b_1) + n(a_1c_2 - a_2c_1) \\ m(a_3a_1 - a_1b_3) + n(a_3c_1 + a_1c_3) & m(a_3b_2 + a_1b_1) + n(a_3c_2 + a_1c_1) \end{pmatrix} \\ &- \begin{pmatrix} m(b_1a_1 + b_2a_3) + n(c_1a_1 + c_2a_3) & m(b_1a_2 - b_2a_1) + n(c_1a_2 + c_2a_1) \\ m(b_3a_1 + b_1a_3) + n(c_3a_1 + c_1a_3) & m(b_3a_2 + b_1a_1) + n(c_3a_2 + c_1a_1) \end{pmatrix} = m[a, b] + n[a, c]. \end{aligned}$$

ii. Next, consider an element bracketed with itself:

$$[a, a] = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix} - \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

iii. Consider the Jacobi identity calculation:

$$
\begin{aligned}
[a,[b,c]]+[b,[c,a]]+[c,[a,b]] &= \left[a, \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}\begin{pmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{pmatrix} - \begin{pmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}\right] \\
&+ \left[b, \begin{pmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix} - \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}\begin{pmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{pmatrix}\right] \\
&+ \left[c, \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix} - \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}\right] \\
&= \left[a, \begin{pmatrix} b_2c_3 - c_2b_3 & 2b_1c_2 - 2b_2c_1 \\ 2b_3c_1 - 2b_1c_3 & b_3c_2 - c_3b_2 \end{pmatrix}\right] + \left[b, \begin{pmatrix} c_2a_3 - a_2c_3 & 2c_1a_2 - 2c_2a_1 \\ 2c_3a_1 - 2c_1a_3 & c_3a_2 - a_3c_2 \end{pmatrix}\right] \\
&+ \left[c, \begin{pmatrix} a_2b_3 - b_2a_3 & 2a_1b_2 - 2a_2b_1 \\ 2a_3b_1 - 2a_1b_3 & a_3b_2 - b_3a_2 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}\begin{pmatrix} b_2c_3 - c_2b_3 & 2b_1c_2 - 2b_2c_1 \\ 2b_3c_1 - 2b_1c_3 & b_3c_2 - c_3b_2 \end{pmatrix} - \begin{pmatrix} b_2c_3 - c_2b_3 & 2b_1c_2 - 2b_2c_1 \\ 2b_3c_1 - 2b_1c_3 & b_3c_2 - c_3b_2 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}\right] \\
&+ \left[\begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}\begin{pmatrix} c_2a_3 - a_2c_3 & 2c_1a_2 - 2c_2a_1 \\ 2c_3a_1 - 2c_1a_3 & c_3a_2 - a_3c_2 \end{pmatrix} - \begin{pmatrix} c_2a_3 - a_2c_3 & 2c_1a_2 - 2c_2a_1 \\ 2c_3a_1 - 2c_1a_3 & c_3a_2 - a_3c_2 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}\right] \\
&+ \left[\begin{pmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{pmatrix}\begin{pmatrix} a_2b_3 - b_2a_3 & 2a_1b_2 - 2a_2b_1 \\ 2a_3b_1 - 2a_1b_3 & a_3b_2 - b_3a_2 \end{pmatrix} - \begin{pmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{pmatrix}\begin{pmatrix} a_2b_3 - b_2a_3 & 2a_1b_2 - 2a_2b_1 \\ 2a_3b_1 - 2a_1b_3 & a_3b_2 - b_3a_2 \end{pmatrix}\right] \\
&= \begin{pmatrix} 2a_2b_3c_1 - 2a_2b_1c_3 - 2a_3b_1c_2 + 2a_3b_2c_1 & 4a_1b_1c_2 - 4a_1b_2c_1 + 2a_2b_3c_2 - 2a_2c_3b_2 \\ 2a_3b_2c_3 - 2a_3c_2b_3 - 4a_1b_3c_1 + 4a_1b_1c_3 & 2a_3b_1c_2 - 2a_3b_2c_1 - 2a_2b_3c_1 + 2a_2b_1c_3 \end{pmatrix} \\
&+ \begin{pmatrix} 2b_2c_3a_1 - 2b_2c_1a_3 - 2b_3c_1a_2 + 2b_3c_2a_1 & 4b_1c_1a_2 - 4b_1c_2a_1 + 2b_2c_3a_2 - 2b_2a_3c_2 \\ 2b_3c_2a_3 - 2b_3a_2c_3 - 4b_1c_3a_1 + 4b_1c_1a_3 & 2b_3c_1a_2 - 2b_3c_2a_1 - 2b_2c_3a_1 + 2b_2c_1a_3 \end{pmatrix} \\
&+ \begin{pmatrix} 2c_2a_3b_1 - 2c_2a_1b_3 - 2c_3a_1b_2 + 2c_3a_2b_1 & 4c_1a_1b_2 - 4c_1a_2b_1 + 2c_2a_3b_2 - 2c_2b_3a_2 \\ 2c_3a_2b_3 - 2c_3b_2a_3 - 4c_1a_3b_1 + 4c_1a_1b_3 & 2c_3a_1b_2 - 2c_3a_2b_1 - 2c_2a_3b_1 + 2c_2a_1b_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}
$$

Thus, we have shown that $\mathfrak{sl}(2)$ satisfies all of the conditions for a Lie algebra.

Since we want to look specifically at the structure of the generalized quaternions, we must consider the Lie algebra of a finite group. Let G be a finite group. Then the Lie algebra of G suggested by Plesken, denoted $\mathcal{L}(G)$, is a linear span of elements $\hat{g} = g - g^{-1} \in G(\mathbb{C})$ for every $g \in G$. The Lie algebra bracket is given by defining

$$[\hat{g}, \hat{h}] = \hat{g}\hat{h} - \hat{h}\hat{g}$$

which extends linearly to all of $\mathcal{L}(G)$. Also note that $\widehat{g^{-1}} = -\hat{g}$ and

$$[\hat{g}, \hat{h}] = \widehat{gh} - \widehat{gh^{-1}} - \widehat{g^{-1}h} + \widehat{g^{-1}h^{-1}}.$$

It follows that $\mathcal{L}(G)$ is closed under the Lie bracket operation. In fact one can show that $\mathcal{L}(G)$ is a Lie algebra.

Since we are describing the Lie algebra structure of the generalized quaternions, it follows that we need to define the generalized quaternions.

Definition 2.2. The generalized quaternion group of order $4n$ is defined as

$$Q_{4n} = \left\langle a, b \mid a^{2n} = b^4 = 1,\ a^n = b^2,\ b^{-1}ab = a^{-1} \right\rangle,$$

where $n \geq 2$.

With this definition, we can consider the specific example of the quaternion group of order eight. The restrictions on generators for Q_8 are as follows:

$$Q_8 = \left\langle a, b \mid a = b^2, b^4 = 1, a^{-1}ba = b^{-1} \right\rangle .$$

We can write the eight elements in the following form: $1, a, a^2, a^3, b, ab, a^2b, a^3b$. Using the Lie algebra defined by Plesken, we have that 1 and a^2 both are their own inverses and a is the inverse of a^3, b is the inverse of a^3b, and ab is the inverse of a^2b, and therefore $\mathcal{L}(Q_8)$ is generated by three elements, $\hat{a}$, $\hat{b}$, and $\widehat{ab}$ where

$$\hat{a} = a - a^3,\ \hat{b} = b - a^3b,\ \text{and}\ \widehat{ab} = ab - a^2b.$$

Hence, $dim(\mathcal{L}(Q_8)) = 3$. Now, let $c = ab$. We have that

$$[\hat{a}, \hat{b}] = 4\hat{c}, \quad [\hat{c}, \hat{a}] = 4\hat{b}, \quad [\hat{b}, \hat{c}] = 4\hat{a}.$$

Note that

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \text{and } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

is a basis for $\mathfrak{sl}(2)$ label them A, B, and C, respectively. We want to show that $\mathcal{L}(Q_8)$ is isomorphic to $\mathfrak{sl}(2)$, so consider the following association:

$$\phi(\hat{a}) \to -2A + 2B = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

$$\phi(\hat{b}) \to 2iA + 2iB = \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix}$$

$$\phi(\hat{c}) \to -2C = \begin{bmatrix} -2i & 0 \\ 0 & 2i \end{bmatrix}.$$

It follows,

$$\begin{aligned}
[\phi(\hat{a}), \phi(\hat{b})] &= \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} - \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -4i & 0 \\ 0 & 4i \end{bmatrix} - \begin{bmatrix} 4i & 0 \\ 0 & -4i \end{bmatrix} = \begin{bmatrix} -8i & 0 \\ 0 & 8i \end{bmatrix} \\
&= 4\phi(\hat{c}) = \phi([\hat{a}, \hat{b}]),
\end{aligned}$$

$$\begin{aligned}
[\phi(\hat{c}), \phi(\hat{a})] &= \begin{bmatrix} -2i & 0 \\ 0 & 2i \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -2i & 0 \\ 0 & 2i \end{bmatrix} \\
&= \begin{bmatrix} 0 & 4i \\ 4i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -4i \\ -4i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 8i \\ 8i & 0 \end{bmatrix} \\
&= 4\phi(\hat{b}) = \phi([\hat{c}, \hat{a}]),
\end{aligned}$$

$$\begin{aligned}
[\phi(\hat{b}), \phi(\hat{c})] &= \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} \begin{bmatrix} -2i & 0 \\ 0 & 2i \end{bmatrix} - \begin{bmatrix} -2i & 0 \\ 0 & 2i \end{bmatrix} \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix} \\
&= 4\phi(\hat{a}) = \phi([\hat{b}, \hat{c}]).
\end{aligned}$$

Hence $\mathcal{L}(Q_8) \cong \mathfrak{sl}(2)$.

In order to describe the associated Lie algebra of a group, it is helpful to define a representation.

Definition 2.3. A **representation** of a finite group G on a finite-dimensional complex vector space V is a group homomorphism $\phi : G \to GL(V)$.

It is important to note that we can encode essential information about representations in a condensed form through character values. In fact, complex representations of finite groups are determined, up to isomorphism, by their characters. Specifically, we define a character as follows.

Definition 2.4. A **character** of V is a complex valued-function defined on G that assigns to each element, $g \in G$, the trace of the linear transformation that g induces on V, that is, $\chi(g) = Tr(\phi(g))$.

With the definition of a representation, we can consider describing the Lie algebra structure of a particular group. In fact, if we can calculate certain character and indicator values, we are able to describe the structure of the Lie algebra for the entire group. That is, we can know what types of Lie algebras, the number of each type of Lie algebra, as well as dimensions of each piece that form the irreducible components of the Lie algebra. This information can be determined through the following theorem.

Theorem 2.5 (Cohen & Taylor, 2007)**.** *The Lie algebra $\mathcal{L}(G)$ admits the decomposition*

$$\mathcal{L}(G) = \bigoplus_{\chi \in \mathfrak{R}} \mathfrak{o}(\chi(1)) \oplus \bigoplus_{\chi \in \mathfrak{Sp}} \mathfrak{sp}(\chi(1)) \oplus \bigoplus_{\chi \in \mathfrak{C}} {}'\mathfrak{gl}(\chi(1))$$

where $\mathfrak{R}$, $\mathfrak{Sp}$, and $\mathfrak{C}$ are the sets of irreducible character of real, sympletic, and complex types, respectively, and where the prime signifies that there is just one summand $\mathfrak{gl}(\chi(1))$ for each pair $\{\chi, \overline{\chi}\}$ from $\mathfrak{C}$.

With this theorem, it is important to know more about the character values and indicator values for a group.

Definition 2.6. The **character table** of G is a $m \times m$ table whose rows correspond to irreducible group representations and columns correspond to the conjugacy classes of each element.

Table 1 is the general from of a character table. Note that each entry of the character table would be of the form $\chi_i(g_m) = Tr(\phi(g_m))$ where each χ_i is a different representation of the group.

In order to use the Theorem 2.5, we also need to know how to calculate the indicator value.

G	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\ldots$
χ_1				
χ_2				
χ_3				
$\vdots$				

Table 1: Character Table

Definition 2.7. Let G be a group with a finite-dimensional continuous complex representation with an associated character χ. Then the **Frobenius-Schur indicator** is defined to be

$$\frac{1}{|G|}\sum_{g\in G}\chi(g^2).$$

Now we can consider the example of Q_8. First, we will calculate one of the indicator values for Q_8. The conjugacy classes for Q_8 are the following: $\{e\}$, $\{a, a^3\}$, $\{a^2\}$, $\{ab, a^3b\}$, $\{b, a^2b\}$. Note that the conjugacy classes are important because every element in a conjugacy class has the same character value, and consequently, the trace of the square of the representation will also be the same for each element. Therefore, we only need to calculate the trace of the square of the representation for one element in each conjugacy class and then multiply this trace by the number of elements in each conjugacy class. Therefore, consider the following matrix representation correspondence:

$$\phi(a)^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ \phi(a)^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \phi(b)^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ \text{and}\ \phi(ab)^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

and our trace values are $Tr(\phi(e)^2) = 2$, $Tr(\phi(a)^2) = -2$, $Tr(\phi(a)^4) = 2$,
$Tr(\phi(ab)^2) = -2$, and $Tr(\phi(b)^2) = -2$.

Thus, our indicator value is the following,

$$\frac{1}{|G|}\sum_{g\in G}\chi(g^2) = \frac{1}{8}(2 + 2(-2) + 2 + 2(-2) + 2(-2)) = -1.$$

The remaining four indicator values for Q_8 are 1; therefore, we have the indicator values for Q_8 arranged as [1, 1, 1, 1, -1]. Note we have the character values for Q_8 in Table 2.
Since by Theorem 2.5, we want $\chi(1)$, we use the values of the first column of the character table. It follows that we have four ones for the indicator values that pair with four ones from the character table, which by Theorem 2.5, give us four cases of $\mathfrak{o}(2)$, that are zero dimensional orthogonal representations. Next we have a -1 indicator value that pairs with a 2 character value. This gives us one case of $\mathfrak{sp}(2)$ which is a three dimensional simplectic representation that is equal to $\mathfrak{sl}(2)$. Thus, we have that $\mathcal{L}(Q_8) = \mathfrak{sp}(2)$, so $\mathcal{L}(Q_8) = \mathfrak{sl}(2)$.

Now consider the example of Q_{16}. The indicator values are [1, 1, 1, 1, 1, -1, -1] for Q_{16}. Note the character values for Q_{16} are in Table 3.

Q_8	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	-1	1
χ_4	1	1	-1	1	-1
χ_5	2	-2	0	0	0

Table 2: Character Table for Q_8

Q_8	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	-1	1	1	1	-1
χ_3	1	-1	1	1	1	-1	1
χ_4	1	1	-1	1	1	-1	-1
χ_5	2	0	0	-2	2	0	0
χ_6	2	0	$\sqrt{2}$	0	-2	0	-$\sqrt{2}$
χ_7	2	0	-$\sqrt{2}$	0	-2	0	$\sqrt{2}$

Table 3: Character Table for Q_{16}

Note that we have four ones for the indicator values that pair with four ones from the character table which by Theorem 2.5 give us four cases of $\mathfrak{o}(1)$, which are zero dimensional orthogonal representation. Next we have a 1 indicator value that pairs with a 2 character value. This gives one case of $\mathfrak{o}(2)$, which is a one dimensional orthogonal representation. Since we are working over the complex numbers, this piece is simply a one dimensional complex representation. Next we have two, -1 indicator values that pair with two, 2 character values. This gives us one case of $\mathfrak{sp}(2)$ which is a three dimensional simplectic representation. That is $\mathcal{L}(Q_{16}) = \mathfrak{sp}(2) \oplus \mathfrak{sp}(2) \oplus \mathfrak{o}(2) \Rightarrow \mathcal{L}(Q_{16}) = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathbb{C}$.

3 Main Results

In order to describe the generalized quaternions completely, we need the character and indicator values. It follows that we must consider matrix representations for the group. Consider the following proposition:

Proposition 3.1. *The matrices* $a = \begin{pmatrix} \omega_n & 0 \\ 0 & \overline{\omega}_n \end{pmatrix}$ *and* $b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$*, where* $\omega_n = e^{\frac{i\pi}{n}}$*, are generators for the generalized quaternion group.*

Proof. Note that $b^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $b^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $b^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Consider

$$a^{2n} = \begin{pmatrix} \omega_n & 0 \\ 0 & \overline{\omega}_n \end{pmatrix}^{2n} = \begin{pmatrix} \omega_n^{2n} & 0 \\ 0 & \overline{\omega}_n^{2n} \end{pmatrix} = \begin{pmatrix} e^{i2\pi} & 0 \\ 0 & \overline{e^{i2\pi}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus we have that the first condition $b^4 = a^{2n} = I$ is satisfied. Note that

$$a^n = \begin{pmatrix} \omega_n & 0 \\ 0 & \overline{\omega}_n \end{pmatrix}^n = \begin{pmatrix} \omega_n^n & 0 \\ 0 & \overline{\omega}_n^n \end{pmatrix} = \begin{pmatrix} e^{i\pi} & 0 \\ 0 & e^{i\pi} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore, we have that the second condition $a^n = b^2$ is satisfied. Now consider

$$b^{-1}ab = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \omega_n & 0 \\ 0 & \overline{\omega}_n \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \overline{\omega}_n \\ -\omega_n & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \overline{\omega}_n & 0 \\ 0 & \omega_n \end{pmatrix}.$$

Thus the third condition $b^{-1}ab = a^{-1}$ is satisfied. Therefore, we have that the matrices a and b are generators for the generalized quaternion group.

□

We get other irreducible representations for the quaternions from the powers of this representation as proven in the following proposition.

Proposition 3.2. *Let $\phi_r : Q_{4n} \to G\mathcal{L}(2, \mathbb{C})$ be given by $a \to \begin{pmatrix} \omega_n & 0 \\ 0 & \overline{\omega}_n \end{pmatrix}^r$ and $b \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then ϕ_r is a group homomorphism for each $\{r \in \mathbb{N} \mid 1 \leq r \leq n-1\}$.*

Proof. Consider $a^j b^k$ where $\{j \in \mathbb{N} \mid 0 \leq j \leq n-1\}$ and $k \in \{0, 1\}$, thus $a^j b^k \in Q_{4n}$ and $a^l b^m$ where $\{l \in \mathbb{N} \mid 0 \leq l \leq n-1\}$ and $m \in \{0, 1\}$, thus $a^l b^m \in Q_{4n}$. We have the following:

$$\phi_r(a^j b^k)\phi_r(a^l b^m) = \begin{pmatrix} \omega_n & 0 \\ 0 & \overline{\omega}_n \end{pmatrix}^{jr} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} \omega_n & 0 \\ 0 & \overline{\omega}_n \end{pmatrix}^{lr} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^m = \phi_r(a^j b^k a^l b^m).$$

□

Note that this representation does not give us all of the irreducible representations. Thus we shall also consider another representation given in the following proposition.

Proposition 3.3. *The matrices $a = \begin{pmatrix} \omega_n & 0 \\ 0 & \overline{\omega}_n \end{pmatrix}$, and $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where $\omega_n = e^{\frac{i2\pi}{n}}$, are generators for the generalized quaternion group.*

Proof. Note that $b^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $b^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $b^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Consider

$$a^{2n} = \begin{pmatrix} \omega_n & 0 \\ 0 & \overline{\omega}_n \end{pmatrix}^{2n} = \begin{pmatrix} \omega_n^{2n} & 0 \\ 0 & \overline{\omega}_n^{2n} \end{pmatrix} = \begin{pmatrix} e^{i4\pi} & 0 \\ 0 & e^{i4\pi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus we have that the first condition $b^4 = a^{2n} = I$ is satisfied. Note that

$$a^n = \begin{pmatrix} \omega_n & 0 \\ 0 & \overline{\omega}_n \end{pmatrix}^n = \begin{pmatrix} \omega_n^n & 0 \\ 0 & \overline{\omega}_n^n \end{pmatrix} = \begin{pmatrix} e^{i2\pi} & 0 \\ 0 & e^{i2\pi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, we have that the second condition $a^n = b^2$ is satisfied. Now consider

$$b^{-1}ab = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \omega_n & 0 \\ 0 & \overline{\omega}_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \overline{\omega}_n \\ \omega_n & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \overline{\omega}_n & 0 \\ 0 & \omega_n \end{pmatrix}.$$

Thus the third condition $b^{-1}ab = a^{-1}$ is satisfied. Therefore, we have that the matrices a and b are generators for the generalized quaternion group.

□

Note that if we use the odd powers of the representation from Proposition 3.1 up to $2n-1$ and use all of the powers of the representation from Proposition 3.3 up to $\frac{n-2}{2}$ we get all of the irreducible representations. Moreover, it will be useful to us to describe the form for the group elements as seen in the following proposition.

Proposition 3.4. *The elements* $\{1, a, a^2, ..., a^{2n-1}, b, ab, a^2b, ..., a^{2n-1}b\}$ *represent the* $4n$ *distinct group elements of* Q_{4n}.

Proof. Let $j, k \in \mathbb{Z}$ such that $0 \leq j \leq 2n-1$ and $0 \leq k \leq 2n-1$. It follows from the definition of the generalized quaternions that $j \neq k$, $a^j \neq a^k$, since a is defined to be of order a^{2n}. Suppose $a^j = a^k b$, thus $a^{j-k} = b$. Since a is a diagonal matrix the powers of a will be a diagonal matrix. However, b is an off diagonal matrix, consequently $a^j \neq a^k b$. Now suppose $a^j b = a^k b$, it follows that $a^j = a^k$, which brings us to the first case. Therefore $a^j b \neq a^k b$. □

Further we can describe the conjugacy classes for the group.

Proposition 3.5. *Each group has* $n+3$ *conjugacy classes, with representatives*

$$\{1\}, \{a^n\}, \bigcup_{1 \leq k < n} \{a^k, a^{-k}\}, \left\{a^k b : \ k \ \textit{even}\right\}, \ \textit{and} \ \left\{a^k b : \ k \ \textit{odd}\right\}.$$

Proof. Let $\{j \in \mathbb{Z} \mid 0 \leq J \leq n-1\}$ and $k \in \{0, 1\}$.

First consider the identity element. It follows that $a^j b^k (1)(a^j b^k)^{-1} = a^j b^k (1)(a^j b^k)^{-1} = 1$. Therefore, 1 forms a conjugacy class.

Next consider the element a^n. There are two cases: $k = 0$ and $k = 1$.

Case 1: Let $k = 0$, then we have $a^j (a^n) a^{-j} = a^{j+n-j} = a^n$.

Case 2: Let $k = 1$ and consider

$$\begin{aligned}
(a^j b)^{-1} a^n a^j b &= b^{-1} a^{-j} a^n a^j b \\
&= b^{-1} abb^{-1} abb^{-1} a ... b^{-1} ab \\
&= a^{-n} \\
&= a^n
\end{aligned}$$

Therefore, there is a conjugacy class consisting of only a^n.

Next we have the element a^j. Let $\{m \in \mathbb{Z} \mid 0 \leq m \leq n-1\}$. Again there are two cases: $k = 0$ and $k = 1$.

Case 1: Let $k = 0$. It follows that $a^m a^j a^{-m} = a^{m+j-m} = a^j$.

Case 2: Let $k = 1$, then we have $(a^m b)^{-1} a^j a^m b = b^{-1} a^{-m} a^j a^m b = b^{-1} a^m b = a^{-j}$.

Therefore, we have that a^j and a^{-j} form a conjugacy class.

Now consider the elements of the form $a^j b$ such that j is even. Let $\{m \in \mathbb{Z} \mid 0 \leq m \leq n-1\}$. Again there are two cases: $k = 0$ and $k = 1$.

Case 1: Let $k = 0$. We have

$$\begin{aligned} a^m a^j b a^{-m} &= a^m a^j b b^{-1} a^m b \\ &= a^m a^j a^m b \\ &= a^{j+2m} b. \end{aligned}$$

Note that $j + 2m$ is even.

Case 2: Let $k = 1$. It follows that

$$\begin{aligned} b^{-1} a^{-m} a^j b a^m b &= b^{-1} a^j b a^m b \\ &= a^{-j+m} a^m b \\ &= a^{-j+2m} b. \end{aligned}$$

Note that $-j + 2m$ is even.

Thus we have that all elements of the form $a^j b$, where j is even form a conjugacy class.

Consider elements of the form $a^j b$ such that j is odd. Let $\{m \in \mathbb{Z} \mid 0 \leq m \leq n-1\}$. Again there are two cases: $k = 0$ and $k = 1$.

Case 1: Let $k = 0$. We have

$$\begin{aligned} a^m a^j b a^{-m} &= a^m a^j b b^{-1} a^m b \\ &= a^m a^j a^m b \\ &= a^{j+2m} b. \end{aligned}$$

Note that $j + 2m$ is odd.

Case 2: Let $k = 1$. Consider the following

$$\begin{aligned} b^{-1}a^{-m}a^j ba^m b &= b^{-1}a^j ba^m b \\ &= a^{-j+m}a^m b \\ &= a^{-j+2m}b. \end{aligned}$$

Note that $-j + 2m$ is odd.

Thus we have that all elements of the form $a^j b$, where j is odd form a conjugacy class.

Based on the Proposition 3.4 we have accounted for all of the elements of Q_{4n}. Now we have that the conjugacy classes are

$$\{1\}, \{a^n\}, \bigcup_{1\le k<n} \left\{a^k, a^{-k}\right\}, \left\{a^k b : \; k \text{ even}\right\}, \text{ and } \left\{a^k b : \; k \text{ odd}\right\}.$$

It follows that a group will have $n-1$ conjugacy classes of the form $\left\{a^k, a^{-k}\right\}$, and we have four other conjugacy classes. Thus there are $n - 1 + 4 = n + 3$ conjugacy classes. □

With this information about the elements, conjugacy classes, and representations we can calculate the indicator value for each type of representation as seen in the following two propositions.

Proposition 3.6. *The indicator value for Q_{4n} of the representations generated by $a = \begin{pmatrix} \omega_n & 0 \\ 0 & \overline{\omega}_n \end{pmatrix}$ and $b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ where $\omega_n = e^{\frac{i\pi}{n}}$, is always -1.*

Proof. Let n be a number of the form 2^j where $j \in \mathbb{N}$ and $j \geq 1$. Since every element in a conjugacy class shares the same character value, it follows that $\chi(g^2)$ would be the same for each element in the same conjugacy classes and by Proposition 3.5 we know what the conjugacy classes for Q_{4n}.

First consider the case $\left\{a^k b : \; k \text{ even}\right\}$.

Since the trace is same for every element in the conjugacy class consider the case where $k = 0$. We have $a^0 b = b$. It follows that $\phi(b)^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Therefore, $Tr(\phi(b)^2) = -2$. Also note that there are n elements in this conjugacy class.

Next consider the class $\left\{a^k b : \; k \text{ odd}\right\}$. Note that

$$\begin{aligned} \phi(a^k b) &= \begin{pmatrix} e^{\frac{k\pi i}{n}} & 0 \\ 0 & e^{\frac{-k\pi i}{n}} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -e^{\frac{k\pi i}{n}} \\ e^{\frac{-k\pi i}{n}} & 0 \end{pmatrix} \text{ and} \\ \phi(a^k b)^2 &= \begin{pmatrix} 0 & -e^{\frac{k\pi i}{n}} \\ e^{\frac{-k\pi i}{n}} & 0 \end{pmatrix}\begin{pmatrix} 0 & -e^{\frac{k\pi i}{n}} \\ e^{\frac{-k\pi i}{n}} & 0 \end{pmatrix} = \begin{pmatrix} -e^0 & 0 \\ 0 & -e^0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

Thus $Tr(\phi(a^k b)^2) = -2$. Also note that there are n elements in this conjugacy class.

Next consider the conjugacy class $\{a, a^{-1}\}$. It follows that

$$\phi(a) = \begin{pmatrix} e^{\frac{\pi i}{n}} & 0 \\ 0 & e^{\frac{-\pi i}{n}} \end{pmatrix} \text{ and}$$
$$\phi(a)^2 = \begin{pmatrix} e^{\frac{\pi i}{n}} & 0 \\ 0 & e^{\frac{-\pi i}{n}} \end{pmatrix} \begin{pmatrix} e^{\frac{\pi i}{n}} & 0 \\ 0 & e^{\frac{-\pi i}{n}} \end{pmatrix} = \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{\frac{-2\pi i}{n}} \end{pmatrix}.$$

Therefore, $Tr(\phi(a)^2) = e^{\frac{2\pi i}{n}} + e^{\frac{-2\pi i}{n}}$ Note for the next conjugacy class $\{a^2, a^{-2}\}$, we get

$$\phi(a) = \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{\frac{-2\pi i}{n}} \end{pmatrix} \text{ and}$$
$$\phi(a)^2 = \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{\frac{-2\pi i}{n}} \end{pmatrix} \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{\frac{-2\pi i}{n}} \end{pmatrix} = \begin{pmatrix} e^{\frac{4\pi i}{n}} & 0 \\ 0 & e^{\frac{-4\pi i}{n}} \end{pmatrix}.$$

Therefore, $Tr(\phi(a)^2) = e^{\frac{4\pi i}{n}} + e^{\frac{-4\pi i}{n}}$ It follows that our conjugacy classes and the corresponding traces are:

$$\begin{array}{cc} \{a, a^{-1}\} & e^{\frac{2\pi i}{n}} + e^{\frac{-2\pi i}{n}} \\ \{a^2, a^{-2}\} & e^{\frac{4\pi i}{n}} + e^{\frac{-4\pi i}{n}} \\ \{a^3, a^{-3}\} & e^{\frac{6\pi i}{n}} + e^{\frac{-6\pi i}{n}} \\ \vdots & \vdots \\ \{a^{2n-2}, a^{-(2n-2)}\} & e^{\frac{(2n-1)\pi i}{n}} + e^{\frac{-(2n-1)\pi i}{n}}. \end{array}$$

We can write the identity as

$$\phi(e) = \begin{pmatrix} e^{\frac{0(\pi i)}{n}} & 0 \\ 0 & e^{\frac{-0(\pi i)}{n}} \end{pmatrix} \text{ and}$$
$$\phi(e)^2 = \begin{pmatrix} e^{\frac{0(\pi i)}{n}} & 0 \\ 0 & e^{\frac{-0(\pi i)}{n}} \end{pmatrix} \begin{pmatrix} e^{\frac{0(\pi i)}{n}} & 0 \\ 0 & e^{\frac{-0(\pi i)}{n}} \end{pmatrix} = \begin{pmatrix} e^{\frac{0(2\pi i)}{n}} & 0 \\ 0 & e^{\frac{-0(2\pi i)}{n}} \end{pmatrix}.$$

Thus $Tr(\phi(e)^2) = e^{\frac{0(2\pi i)}{n}} + e^{\frac{0(-2\pi i)}{n}}$.

We also have the conjugacy class $\{a^n\}$. That is

$$\phi(a^n) = \begin{pmatrix} e^{\frac{n\pi i}{n}} & 0 \\ 0 & e^{\frac{-n\pi i}{n}} \end{pmatrix} \text{ and}$$
$$\phi(a^n)^2 = \begin{pmatrix} e^{\frac{n\pi i}{n}} & 0 \\ 0 & e^{\frac{-n\pi i}{n}} \end{pmatrix} \begin{pmatrix} e^{\frac{n\pi i}{n}} & 0 \\ 0 & e^{\frac{-n\pi i}{n}} \end{pmatrix} = \begin{pmatrix} e^{\frac{2n\pi i}{n}} & 0 \\ 0 & e^{\frac{-2n\pi i}{n}} \end{pmatrix}.$$

Thus $Tr(\phi(a^n)^2) = e^{\frac{2n\pi i}{n}} + e^{\frac{-2n\pi i}{n}}$.

Now consider the following sum:

$$e^{\frac{0(2\pi i)}{n}} + e^{\frac{0(-2\pi i)}{n}} + e^{\frac{2\pi i}{n}} + e^{\frac{-2\pi i}{n}} + e^{\frac{4\pi i}{n}} + e^{\frac{-4\pi i}{n}} + e^{\frac{6\pi i}{n}} + e^{\frac{-6\pi i}{n}} + ... + e^{\frac{(2n-2)\pi i}{n}} + e^{\frac{-(2n-2)\pi i}{n}} + e^{\frac{2n\pi i}{n}} + e^{\frac{-2n\pi i}{n}}$$

$$= \sum_{r=0}^{n} e^{\frac{2\pi i r}{n}} + e^{\frac{-2\pi i r}{n}} = \sum_{r=0}^{n} \omega^r + \omega^{-r} = 0 + 0 = 0.$$

Thus when calculating the indicator value we have

$$\frac{1}{|g|}\sum_{g\in G}\chi(g^2) = \frac{1}{4n}(0 - 2n - 2n) = \frac{-4n}{n} = -1.$$

□

Proposition 3.7. *The indicator value for* Q_{4n} *of the representations generated by* $a = \begin{pmatrix} \omega_n & 0 \\ 0 & \overline{\omega}_n \end{pmatrix}$ *and* $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$*, where* $\omega_n = e^{\frac{2\pi i}{n}}$*, is always -1.*

Proof. Let n be a number of the form 2^j, where $j \in \mathbb{N}$ and $j \geq 1$. Since every element in a conjugacy class shares the same character value, it follows that $\chi(g^2)$ would be the same for each element in the same conjugacy classes and by Proposition 3.5 we know what the conjugacy classes for Q_{4n}.

First consider the class $\{a^k b : \ k \text{ even}\}$.

Since the trace is same for every element in the conjugacy class consider the case where $k = 0$. We have $a^0 b = b$. It follows that $\phi(b)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore, $Tr(\phi(b)^2) = 2$. Also note that there are n elements in this conjugacy class.

Next consider the class $\{a^k b : \ k \text{ odd}\}$. Note that

$$\phi(a^k b) = \begin{pmatrix} e^{\frac{2k\pi i}{n}} & 0 \\ 0 & e^{\frac{-2k\pi i}{n}} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{\frac{2k\pi i}{n}} \\ e^{\frac{-2k\pi i}{n}} & 0 \end{pmatrix} \text{ and}$$
$$\phi(a^k b)^2 = \begin{pmatrix} 0 & e^{\frac{2k\pi i}{n}} \\ e^{\frac{-2k\pi i}{n}} & 0 \end{pmatrix}\begin{pmatrix} 0 & -e^{\frac{2k\pi i}{n}} \\ e^{\frac{-2k\pi i}{n}} & 0 \end{pmatrix} = \begin{pmatrix} e^0 & 0 \\ 0 & e^0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $Tr(\phi(a^k b)^2) = 2$. Also note that there are n elements in this conjugacy class.

Next consider the conjugacy class $\{a, a^{-1}\}$. It follows that

$$\phi(a) = \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{\frac{-2\pi i}{n}} \end{pmatrix} \text{ and}$$
$$\phi(a)^2 = \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{\frac{-2\pi i}{n}} \end{pmatrix} \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{\frac{-2\pi i}{n}} \end{pmatrix} = \begin{pmatrix} e^{\frac{4\pi i}{n}} & 0 \\ 0 & e^{\frac{-4\pi i}{n}} \end{pmatrix}.$$

Therefore, $Tr(\phi(a)^2) = e^{\frac{4\pi i}{n}} + e^{\frac{-4\pi i}{n}}$ Note for the next conjugacy class $\{a^2, a^{-2}\}$, we get

$$\phi(a) = \begin{pmatrix} e^{\frac{4\pi i}{n}} & 0 \\ 0 & e^{\frac{-4\pi i}{n}} \end{pmatrix} \text{ and}$$
$$\phi(a)^2 = \begin{pmatrix} e^{\frac{4\pi i}{n}} & 0 \\ 0 & e^{\frac{-4\pi i}{n}} \end{pmatrix} \begin{pmatrix} e^{\frac{4\pi i}{n}} & 0 \\ 0 & e^{\frac{-4\pi i}{n}} \end{pmatrix} = \begin{pmatrix} e^{\frac{16\pi i}{n}} & 0 \\ 0 & e^{\frac{-16\pi i}{n}} \end{pmatrix}.$$

Therefore, $Tr(\phi(a)^2) = e^{\frac{16\pi i}{n}} + e^{\frac{-16\pi i}{n}}$ It follows that our conjugacy classes and the corresponding traces are:

$$\begin{array}{cc} \{a, a^{-1}\} & e^{\frac{4\pi i}{n}} + e^{\frac{-4\pi i}{n}} \\ \{a^2, a^{-2}\} & e^{\frac{16\pi i}{n}} + e^{\frac{-16\pi i}{n}} \\ \{a^3, a^{-3}\} & e^{\frac{36\pi i}{n}} + e^{\frac{-36\pi i}{n}} \\ \vdots & \vdots \\ \{a^{2n-2}, a^{-(2n-2)}\} & e^{\frac{(2n-1)2\pi i}{n}} + e^{\frac{-(2n-1)2\pi i}{n}}. \end{array}$$

We can write the identity as

$$\phi(e) = \begin{pmatrix} e^{\frac{0(2\pi i)}{n}} & 0 \\ 0 & e^{\frac{-0(2\pi i)}{n}} \end{pmatrix} \text{ and}$$
$$\phi(e)^2 = \begin{pmatrix} e^{\frac{0(2\pi i)}{n}} & 0 \\ 0 & e^{\frac{-0(2\pi i)}{n}} \end{pmatrix} \begin{pmatrix} e^{\frac{0(2\pi i)}{n}} & 0 \\ 0 & e^{\frac{-0(2\pi i)}{n}} \end{pmatrix} = \begin{pmatrix} e^{\frac{0(4\pi i)}{n}} & 0 \\ 0 & e^{\frac{-0(4\pi i)}{n}} \end{pmatrix}.$$

Thus $Tr(\phi(e)^2) = e^{\frac{0(4\pi i)}{n}} + e^{\frac{0(-4\pi i)}{n}}$.

We also have the conjugacy class $\{a^n\}$. That is

$$\phi(a^n) = \begin{pmatrix} e^{\frac{2n\pi i}{n}} & 0 \\ 0 & e^{\frac{-2n\pi i}{n}} \end{pmatrix} \text{ and}$$
$$\phi(a^n)^2 = \begin{pmatrix} e^{\frac{2n\pi i}{n}} & 0 \\ 0 & e^{\frac{-2n\pi i}{n}} \end{pmatrix} \begin{pmatrix} e^{\frac{2n\pi i}{n}} & 0 \\ 0 & e^{\frac{-2n\pi i}{n}} \end{pmatrix} = \begin{pmatrix} e^{\frac{4n\pi i}{n}} & 0 \\ 0 & e^{\frac{-4n\pi i}{n}} \end{pmatrix}.$$

Thus $Tr(\phi(a^n)^2) = e^{\frac{4n\pi i}{n}} + e^{\frac{-4n\pi i}{n}}$.

Now consider the following sum:

$$e^{\frac{0(4\pi i)}{n}} + e^{\frac{0(-4\pi i)}{n}} + e^{\frac{4\pi i}{n}} + e^{\frac{-4\pi i}{n}} + e^{\frac{16\pi i}{n}} + e^{\frac{-16\pi i}{n}} + e^{\frac{36\pi i}{n}} + e^{\frac{-36\pi i}{n}} + ... + e^{\frac{(2n-2)2\pi i}{n}} + e^{\frac{-(2n-2)2\pi i}{n}}$$
$$+ e^{\frac{4n\pi i}{n}} + e^{\frac{-4n\pi i}{n}} = \sum_{r=0}^{n} e^{\frac{4\pi ir}{n}} + e^{\frac{-4\pi ir}{n}} = \sum_{r=0}^{n} \omega^{2r} + \omega^{-2r} = 0 + 0 = 0.$$

Thus when calculating the indicator value we have

$$\frac{1}{|g|} \sum_{g \in G} \chi(g^2) = \frac{1}{4n}(0 + 2n + 2n) = \frac{4n}{n} = 1.$$

□

Recall from our earlier examples that $\mathcal{L}(Q_8) = \mathfrak{sp}(2) \Rightarrow \mathcal{L}(Q_8) = \mathfrak{sl}(2)$ and $\mathcal{L}(Q_{16}) = \mathfrak{sp}(2) \oplus \mathfrak{sp}(2) \oplus \mathfrak{o}(2) \Rightarrow \mathcal{L}(Q_{16}) = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathbb{C}$. In looking at these examples and examples of greater n values we see a pattern, which we describe with the following theorem.

Theorem 3.8. *Let $n = 2^j$, where $j \in \mathbb{N}$. The Lie algebra of Q_{4n} is*

$$\mathcal{L}(Q_{4n}) = \bigoplus_{k=1}^{2^{j-1}} \mathfrak{sp}(2) \oplus \bigoplus_{k=1}^{2^{j-1}-1} \mathfrak{o}(2)$$
$$= \bigoplus_{k=1}^{2^{j-1}} \mathfrak{sl}(2) \oplus \bigoplus_{k=1}^{2^{j-1}-1} \mathbb{C}.$$

Proof. We know that the odd powers of representations from Proposition 3.1 from 1 to $2n-1$ and that all powers of the representations from 1 to $\frac{n-2}{2}$ from Proposition 3.3 form all the irreducible representations for Q_n. By Proposition 3.6 we have that all of the representations from Proposition 3.1 have an indicator of -1. Note that the taking the odd powers from 1 to $2n-1$ gives us $\frac{n}{2}$ representations. However, since n is of the form 2^j, $j \in \mathbb{N}$ we have 2^{j-1} indicator values of -1.

Similarly by Proposition 3.7 we have that all of the representations from Proposition 3.3 have an indicator of 1. Note that since we have $\frac{n-2}{2} = \frac{2^j-2}{2} = 2^{j-1} - 1$. Hence there are $2^{j-1} - 1$ indicator values of 1. Note that we always have 4 one dimensional representations. The identity for these representations is simply the number 1. In considering the indicator values for these representations, it is know that for all four the indicator value is 1.

Note that for all the matrix representations $\chi(1)$ is χ of the identity element which will be the trace of the identity matrix. All of our matrix representations are two dimensional, we have that our character value is $Tr(\chi(1)) = 2$. Since we have 2^{j-1} representations from the representations described in Proposition 3.1 and $2^{j-1} - 1$ representations from the representations described in 3.3, we have a total of $2^{j-1} + 2^{j-1} - 1 = 2^j - 1 = n - 1$ character values that are 2. For all four of the one dimensional representations our character value will be 1 since $Tr(1) = 1$.

It follows that using Theorem 2.5 that the four indicator values of 1 from the one dimensional representations pair with the four Character values or 1, to give four cases of $\mathfrak{o}(1)$ which are zero dimensional representations. From the two dimensional matrix representations we 2^{j-1} indicator values of -1, that pair with 2^{j-1} character values of 2, which from Theorem 2.5 gives us 2^{j-1} cases of $\mathfrak{sp}(2) = \mathfrak{sl}(2)$. From the remaining two dimensional matrix representations we have $2^{j-1} - 1$ indicator values of 1 which pair with the remaining $2^{j-1} - 1$ character values which are all 2s, thus we have $2^{j-1} - 1$ cases of $\mathfrak{o}(2) = \mathbb{C}$.

□

References

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