Fractional Calculus and its Connection to the Tautochrone

by

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Abstract

Despite its brief mention in a letter written during the early days of classical calculus, Fractional Calculus remains a relatively untapped field. With most major contributions occurring in the last one-hundred years. In this paper, we will examine the fundamental aspects of Fractional Calculus and demonstrate how the modern definitions of the Fractional Integral naturally arise from solving the classic Tautochrone problem: finding a curve such that the time it takes an object to fall along this path is independent of its initial position. We then consider a generalization of the tautochrone by investigating the case when time is dependent on initial position. The result is a corollary building off the work published by Muñoz and Fernández-Anaya. We will also examine the Mittag-Leffler function, and how it arises in the solution to Abels Integral Equation of the second kind.

Acknowledgments

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Contents

1 Introduction and Fractional Calculus Fundamentals

In 1695, de l'Hospital wrote a letter to Leibniz asking how to interpret the newly invented derivative when n was not a natural number. Leibniz's response essentially told l'Hospital that it would be a problem for future generations of mathematicians. While many mathematicians explored fractional calculus over the centuries, including Fourier, Euler, Laplace, and Riemann, it did not develop the significance it has today until the 20th century [8], when engineers began to notice how useful it was in many problems such as modeling stress and visco-elasticity. In fact, the Caputo operator we explore in this paper was not introduced until 1967 [9], and in 1996 Kolowankar reformulated the Riemann-Liouville fractional derivative in order to differentiate no-where differentitable fractal functions [8]. The plethora of novel applications and unanswered questions in fractional calculus makes it one of the most interesting fields of mathematics today. We will begin our exploration of fractional calculus by examining the Riemann-Liouville Fractional Integral.

Definition 1.1: Let $\gamma \in \mathbb{R}^+$, then the γth order *Riemann-Liouville integral*, denoted J_a^γ, is given by

$$J_a^\gamma f(x) := \frac{1}{\Gamma(\gamma)} \int_a^x \frac{f(t)}{(x-t)^{1-\gamma}} \, dt. \tag{1}$$

Example 1.1: We will calculate the γth order integral of the function $(x-a)^\beta$ with $\beta > -1$ and $\gamma > 0$. Using definition 1.1, it follows that

$$J_a^\gamma \left[(x-a)^\beta\right] = \frac{1}{\Gamma(\gamma)} \int_a^x (t-a)^\beta \, (x-t)^{\gamma-1} \, dt. \tag{2}$$

Let $t = a + s(x-a)$, then $dt = (x-a)\, ds$. Note that when $t = a$, $s = 0$, and when $t = x$ we find that $s = 1$. Therefore (2) becomes

$$\frac{1}{\Gamma(\gamma)} \int_0^1 (s(x-a))^\beta (x-a-s(x-a))^{\gamma-1} (x-a) \, ds. \tag{3}$$

Combining like terms we arrive at

$$\frac{1}{\Gamma(\gamma)} (x-a)^{\beta+\gamma} \int_0^1 s^\beta (1-s)^{\gamma-1} \, ds. \tag{4}$$

The integral in (4) is in fact Euler's Beta Function[5], defined to be

$$\int_0^1 s^\beta (1-s)^{\gamma-1} \, ds = \frac{\Gamma(\beta+1)\Gamma(\gamma)}{\Gamma(\beta+\gamma+1)}. \tag{5}$$

Note that β and γ must be positive real numbers in order to apply Euler's Beta Function. Inserting

(5) into (4) we conclude that

$$J_a^\gamma\left[(x-a)^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma+1)}(x-a)^{\beta+\gamma}. \tag{6}$$

Where there is an integral, there must be a derivative hiding somewhere. An interesting characteristic of fractional calculus is the development of multiple definitions for the γth order derivative. In this paper we will focus on the Riemann-Liouville form and the Caputo form.

Definition 1.2: Let $\gamma \in \mathbb{R}^+$, and $m = \lceil \gamma \rceil$. Where $\lceil \gamma \rceil$ is the ceiling funtion defined as the smallest integer greater than γ. Then the *Riemann-Liouville fractional derivative*, denoted D_a^γ, is given by

$$D_a^\gamma f := D^m J^{m-\gamma} f. \tag{7}$$

From this definition we may derive a formula similar to the Riemann-Liouville integral:

Theorem 1.1: For $f \in A^1[a,b]$, where $A^m[a,b]$ is the set of functions with absolutely continuous derivatives of order $m-1$, and $0 < \gamma < 1$, the Riemann-Liouville derivative my be computed by:

$$D_a^\gamma f(x) = \frac{1}{\Gamma(1-\gamma)}\left[\frac{f(a)}{(x-a)^\gamma} + \int_a^x f'(t)(x-t)^{-\gamma}\, dt\right]. \tag{8}$$

We follow the proof presented in [5].

Proof. Let $0 < \gamma < 1$. By definition 1.2,

$$\begin{aligned}
D_a^\gamma f &= \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_a^x f(t)(x-t)^{-\gamma}\, dt \\
&= \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_a^x \left[f(a) + \int_a^t f(u) du \right] (x-t)^{-\gamma}\, dt \\
&= \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \left[f(a) \int_a^x (x-t)^{-\gamma} dt + \int_a^x \int_a^t f'(u)(x-t)^{-\gamma} du\, dt \right] \\
&= \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \left[\frac{f(a)(x-a)^{1-\gamma}}{(1-\gamma)} + \int_a^x \int_a^t f'(u)(x-t)^{-\gamma} du\, dt \right] \\
&= \frac{1}{\Gamma(1-\gamma)} \left[\frac{f(a)}{(x-a)^\gamma} + \frac{d}{dx} \int_a^x \int_a^t f'(u)(x-t)^{-\gamma} du\, dt \right].
\end{aligned}$$

Applying Fubini's Theorem,

$$\begin{aligned}
&= \frac{1}{\Gamma(1-\gamma)} \left[\frac{f(a)}{(x-a)^\gamma} + \frac{d}{dx} \int_a^x \int_u^x f'(u)(x-t)^{-\gamma} dt\, du \right] \\
&= \frac{1}{\Gamma(1-\gamma)} \left[\frac{f(a)}{(x-a)^\gamma} + \frac{d}{dx} \int_a^x f'(u) \frac{(x-u)^{1-\gamma}}{1-\gamma}\, du \right] \\
&= \frac{1}{\Gamma(1-\gamma)} \left[\frac{f(a)}{(x-a)^\gamma} + \int_a^x f'(u) \frac{\partial}{\partial x} \frac{(x-u)^{1-\gamma}}{1-\gamma}\, du \right] \\
&= \frac{1}{\Gamma(1-\gamma)} \left[\frac{f(a)}{(x-a)^\gamma} + \int_a^x f'(u)(x-u)^{-\gamma}\, du \right].
\end{aligned}$$

Thus $D_a^\gamma f = \frac{1}{\Gamma(1-\gamma)} \left[\frac{f(a)}{(x-a)^\gamma} + \int_a^x f'(u)(x-u)^{-\gamma}\, du \right]$ as desired. □

Using (7) we can calculate the γth order derivative of $(x-a)^\beta$. By Definition 1.2:

$$\begin{aligned}
D^\gamma (x-a)^\beta &= D^m J^{m-\gamma} (x-a)^\beta \\
&= D^m \left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+m-\gamma+1)} (x-a)^{\beta+m-\gamma} \right] \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta+m-\gamma+1)} \frac{\Gamma(\beta+m-\gamma+1)}{\Gamma(\beta-\gamma+1)} (x-a)^{\beta-\gamma} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)} (x-a)^{\beta-\gamma}. \qquad (9)
\end{aligned}$$

One way to see if this formula makes any sense is to see what happens in the classical case. In particular, we can consider what happens when $\gamma = 1$. Then, according to (9),

$$D^1(x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta)}(x-a)^{\beta-1}$$
$$= \beta(x-a)^{\beta-1}.$$

Which is exactly what we would expect.
Perhaps a more interesting example is the case of the γth derivative of a constant real function C. First note that:

$$J^{m-\gamma}\,C = \frac{C}{\Gamma(m-\gamma)}\int_a^x (x-t)^{m-\gamma-1}\,dt$$
$$= \frac{C(x-a)^{m-\gamma}}{\Gamma(m-\gamma+1)}. \tag{10}$$

Now we take the mth derivative of (10):

$$D^m\left[\frac{C(x-a)^{m-\gamma}}{\Gamma(m-\gamma+1)}\right] = \frac{C}{\Gamma(1-\gamma)}(x-a)^{-\gamma}. \tag{11}$$

We can now consider, for instance, the half-dervative. That is, when $\gamma = 1/2$ and $m = 1$.

$$D^{1/2}\,C = \frac{C}{\Gamma(1-\frac{1}{2})}(x-a)^{-1/2}$$
$$= \frac{C}{\sqrt{\pi}}(x-a)^{-1/2}. \tag{12}$$

So when taking the fractional derivative of a constant function we obtain a nonzero result. We can compare results such as this to another defintion of fractional derivatives: the Caputo Derivative. We will follow the definition found in [5].
Definition 1.3: Let $\gamma \geq 0$ and $m = \lceil \gamma \rceil$. Then we define the *Caputo fractional derivative*, denoted ${}^cD_a^\gamma$, by

$${}^cD_a^\gamma f := J_a^{m-\gamma} D^m f. \tag{13}$$

To demonstrate the fundamental differences between the Riemann-Liouville and Caputo derivatives, we can revisit the derivative of a real constant function C. Recall that we found the Riemann-Liouville derivative of C to be,

$$\frac{C}{\sqrt{\pi}}(x-a)^{-1/2}. \tag{14}$$

for $\gamma = 1/2$. Observe that when we take the Caputo derivative of C with $\gamma = 1/2$ we obtain:

$$\begin{aligned} J_a^{1/2} D^1 C &= J_a^{1/2}\, 0 \\ &= 0. \end{aligned} \tag{15}$$

This is what we would expect from the classical derivative operator. A natural question to ask now is when does the Rimeann-Liouville derivative equal the Caputo derivative? To answer this question we will explore the following theorem[5]:

Theorem 1.2: Let $\gamma > 0$, and $m = \lceil \gamma \rceil$. Assume $f \in A^m[a,b]$. Then the γth Caputo derivative, denoted ${}^cD_a^\gamma$, is given by

$$ {}^cD_a^\gamma f = D_a^\gamma \big[f - T_{m-1}[f;a] \big] \tag{16}$$

where T_{m-1} denotes the Taylor Polynomial of degree $m-1$ for f, and D_a^γ is the Riemann-Liouville derivative.

We will first observe what happens in the classical case, that is when $\gamma \in \mathbb{N}$. Note the following,

$$\begin{aligned} {}^cD_a^\gamma f &= D_a^\gamma \big[f - T_{m-1}[f;a] \big] \\ &= D_a^\gamma\, f - D_a^\gamma\, T_{m-1}[f;a] \\ &= D_a^\gamma\, f. \end{aligned} \tag{17}$$

So we find that, as expected, the Riemann-Liouville derivative equals the caputo in the classical case, but what about the fractional case? The following corollary form Theorem 1.2 is the key:
Corollary:

$$ {}^cD_a^\gamma f = D_a^\gamma\, f - \sum_{k=0}^{m-1} \frac{D^k f(a)}{\Gamma(k-\gamma+1)} (x-a)^{k-\gamma}. \tag{18}$$

Proof. Note the following,

$$\begin{aligned} D_a^\gamma \big[T_{m-1} \big] &= D_a^\gamma \Big[\sum_{k=0}^{m-1} \frac{D^k f(a)}{\Gamma(k+1)} (x-a)^k \Big] \\ &= \sum_{k=0}^{m-1} \frac{D^k f(a)}{\Gamma(k-\gamma+1)} (x-a)^{k-\gamma}. \end{aligned} \tag{19}$$

Where (19) follows from (9), and consequently

$$^{c}D_a^{\gamma} f = D_a^{\gamma} f - \sum_{k=0}^{m-1} \frac{D^k f(a)}{\Gamma(k-\gamma+1)}(x-a)^{k-\gamma}.$$

□

Examining this corollary, it becomes clear that $^{c}D_a^{\gamma} f = D_a^{\gamma} f$ if and only if,

$$\sum_{k=0}^{m-1} \frac{D^k f(a)}{\Gamma(k-n+1)}(x-a)^{k-n} \equiv 0. \tag{20}$$

Which means that $D^k f(a) = 0$ for all $k = 0, 1, ..., m-1$.

The Caputo derivative tends to behave more like the classical derivative than the Riemann-Liouville. This can be seen by examining the Laplace Transform of each derivative.

Recall the Laplace Transform of the classical derivative $D^n f$ for $n \in \mathbb{N}$,

$$\mathscr{L}\{D^n f\}(s) = s^n \mathscr{L}\{f\}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{n-1}(0). \tag{21}$$

Let $m = \lceil \gamma \rceil$ and $0 < \gamma < 1$. Consider the Laplace Transform of Riemann-Liouville Derivative,

$$\begin{aligned}
\mathscr{L}\{D^{\gamma} f\}(s) &= \mathscr{L}\{D^1 J^{1-\gamma} f\}(s) \\
&= s\mathscr{L}\{J^{1-\gamma} f\}(s) \\
&= \frac{s}{\Gamma(1-\gamma)} \mathscr{L}\{f * t^{-\gamma}\}(s) \\
&= \frac{s}{\Gamma(1-\gamma)} \mathscr{L}\{f\}(s) \mathscr{L}\{t^{-\gamma}\}(s) \\
&= \frac{s}{\Gamma(1-\gamma)} F(s) \frac{\Gamma(1-\gamma)}{s^{1-\gamma}} \\
&= s^{\gamma} F(s). \qquad (22)
\end{aligned}$$

Next observe the Laplace Transform of the Caputo Derivative under the same parameters for m and γ,

$$\begin{aligned}
\mathscr{L}\{{}^{c}D^{\gamma}f\}(s) &= \mathscr{L}\{J^{1-\gamma}D^{1}f\}(s) \\
&= \mathscr{L}\{J^{1-\gamma}f'\}(s) \\
&= \frac{1}{\Gamma(1-\gamma)}\mathscr{L}\{f' * t^{-\gamma}\}(s) \\
&= \frac{1}{\Gamma(1-\gamma)}\mathscr{L}\{f'\}(s)\mathscr{L}\{t^{-\gamma}\}(s) \\
&= \frac{1}{\Gamma(1-\gamma)}\left[sF(s) - f(0)\right]\frac{\Gamma(1-\gamma)}{s^{1-\gamma}} \\
&= \frac{sF(s) - f(0)}{s^{1-\gamma}} \\
&= s^{\gamma}F(s) - s^{\gamma-1}f(0). \qquad (23)
\end{aligned}$$

Comparing (22) and (23) to (21), we find that (23) follows the pattern found in the classical case more closely than the Riemann-Liouville transform in (22).

2 The Mittag-Leffler Fucntion

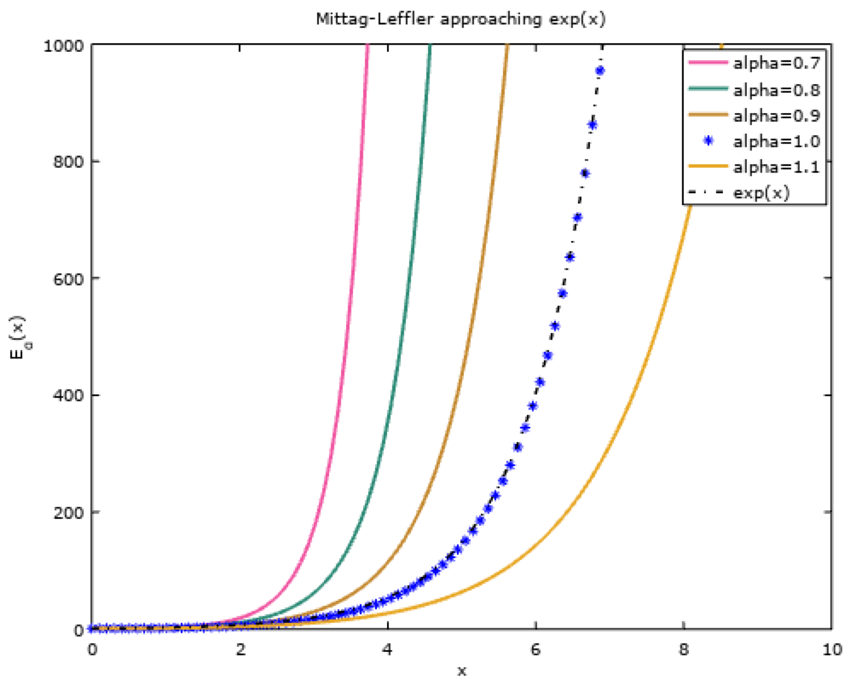
Let $n > 0$, the *Mittag-Leffler function*, denoted $E_\alpha(z)$, is defined by

$$E_\alpha(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\alpha + 1)}. \tag{24}$$

In particular, (24) is the single parameter Mittag-Leffer function. In some cases, it is useful to include a second parameter.

$$E_{\alpha,\beta}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\alpha + \beta)}. \tag{25}$$

This is called the *two-parameter Mittag-Leffler function.* In the single parameter case, by varying α and z, we can obtain many familiar functions. When $z = x$, with $x \in \mathbb{R}$, and $\alpha = 1$, we obtain the exponential function as seen in Figure 2.1



We may also recover $1/(1-x)$ by letting $\alpha = 0$ as seen in Figure 2.2:

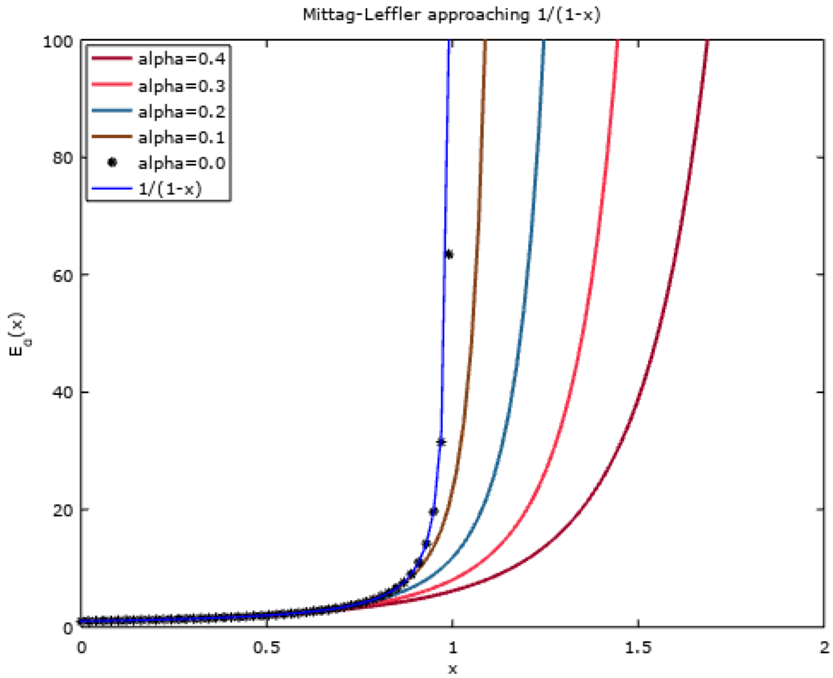
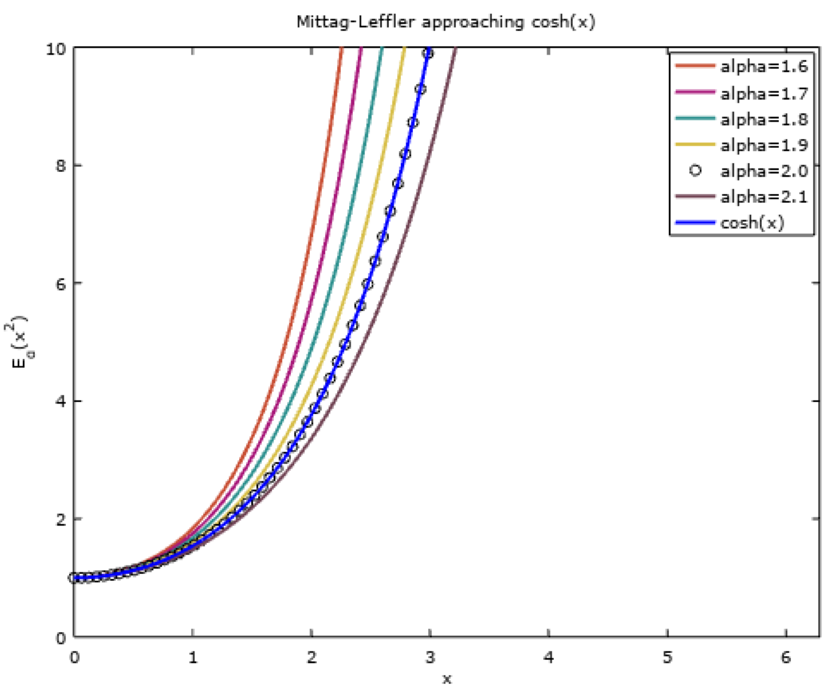
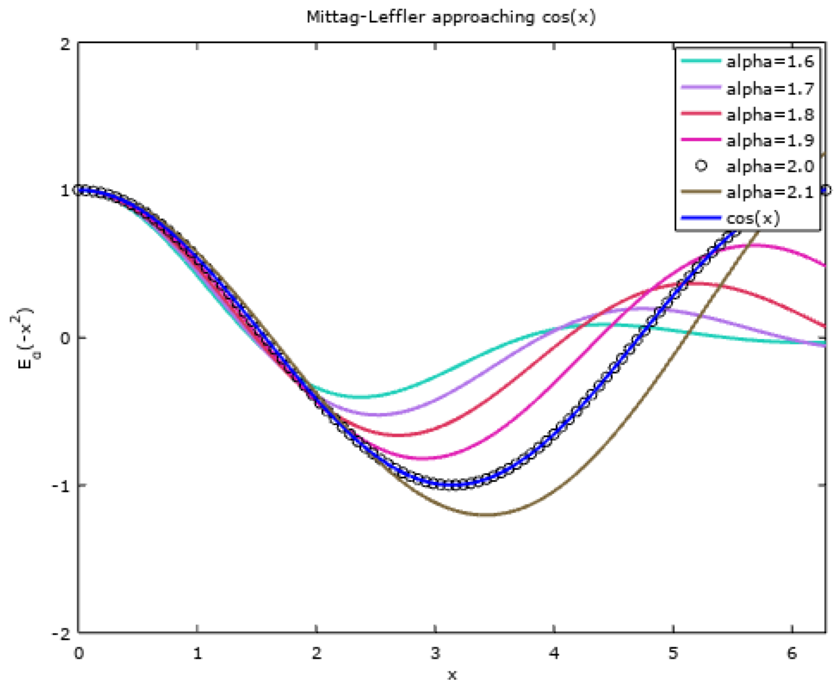


Figure 2.3 shows how $\cosh(x)$ appears for $z = x^2$ and $\alpha = 2$:



and Figure 2.4 demonstrates how $\cos(x)$ appears when $\alpha = 2$ and $z = -x^2$.



As we have seen the Mittag-Leffler function contains the exponential function, therefore it may not be a surprise that the Mittag-Leffler takes the role that e^x plays in classical differential equations in the world of fractional differential equations. We may consider the following FDE:

$$ {}^cD^\gamma y + \lambda y = 0. \tag{26} $$

The Mittag-Leffler function allows us to find a general solution to (26) for any $\gamma \in \mathbb{R}^+$. We will attack this problem using the Laplace transform. Observe the following,

$$
\begin{aligned}
\mathscr{L}\{{}^cD^\gamma y + \lambda y\}(s) &= \mathscr{L}\{J^{m-\gamma}D^m y + \lambda y\}(s) \\
&= \mathscr{L}\{J^{m-\gamma}y^{(m)} + \lambda y\}(s) \\
&= \frac{1}{\Gamma(m-\gamma)}\mathscr{L}\{y^{(m)} * t^{m-\gamma-1}\}(s) + \lambda Y(s) \\
&= \frac{1}{\Gamma(m-\gamma)}\left[s^m Y(s) - s^{m-1}y(0) - \cdots - sy^{m-2} - y^{m-1}\right]\frac{\Gamma(m-\gamma)}{s^{m-\gamma}} + \lambda Y(s) \\
&= s^\gamma Y(s) - s^{\gamma-1}y(0) - \cdots - s^{\gamma-m+1}y^{(m-2)} - s^{\gamma-m}y^{(m-1)} + \lambda Y(s) \\
&= 0.
\end{aligned} \tag{27}
$$

Grouping the $Y(s)$ terms,

$$
\begin{aligned}
Y(s) &= \frac{s^{\gamma-1}y(0) + s^{\gamma-2}y'(0) + \cdots + s^{\gamma-m+1}y^{(m-2)} + s^{\gamma-m}y^{(m-1)}}{s^\gamma + \lambda} \\
&= s^{-1}y(0)\frac{s^\gamma}{s^\gamma+\lambda} + s^{-2}y'(0)\frac{s^\gamma}{s^\gamma+\lambda} + \cdots + s^{-m+1}y^{(m-2)}\frac{s^\gamma}{s^\gamma+\lambda} + s^{-m}y^{(m-1)}\frac{s^\gamma}{s^\gamma+\lambda} \\
&= s^{-1}y(0)\sum_{j=0}^{\infty}(-\lambda s^{-\gamma})^j + s^{-2}y'(0)\sum_{j=0}^{\infty}(-\lambda s^{-\gamma})^j + \cdots + s^{-m+1}y^{(m-2)}\sum_{j=0}^{\infty}(-\lambda s^{-\gamma})^j + s^{-m}y^{(m-1)}\sum_{j=0}^{\infty}(-\lambda s^{-\gamma})^j \\
&= y(0)\sum_{j=0}^{\infty} -\lambda^j s^{-j\gamma-1} + y'(0)\sum_{j=0}^{\infty}\lambda^j s^{-j\gamma-2} + \cdots + y^{(m-2)}\sum_{j=0}^{\infty} -\lambda^j s^{-j\gamma-m+1} + y^{(m-1)}\sum_{j=0}^{\infty} -\lambda^j s^{-j\gamma-m}
\end{aligned} \tag{28}
$$

We will consider the last term for a moment,

$$
\begin{aligned}
y^{(m-1)}\sum_{j=0}^{\infty} -\lambda^j s^{-j\gamma-m} &= y^{(m-1)}\sum_{j=0}^{\infty} -\lambda^j s^{-j\gamma-m}\frac{\Gamma(j\gamma+m)}{\Gamma(j\gamma+m)} \\
&= y^{(m-1)}\sum_{j=0}^{\infty}\frac{-\lambda^j}{\Gamma(j\gamma+m)}\frac{\Gamma(j\gamma+m)}{s^{j\gamma+m}}
\end{aligned} \tag{29}
$$

Applying the inverse Laplace yields,

$$
\begin{aligned}
\mathscr{L}^{-1}\Big\{y^{(m-1)}\sum_{j=0}^{\infty} -\lambda^j s^{-j\gamma-m}\Big\}(x) &= y^{(m-1)}\sum_{j=0}^{\infty}\frac{-\lambda^j}{\Gamma(j\gamma+m)}x^{j\gamma+m-1} \\
&= y^{(m-1)}x^{m-1}E_{\gamma,m}(-\lambda x^\gamma).
\end{aligned} \tag{30}
$$

We may do this for each term in (28), therefore

$$
\begin{aligned}
f(x) &= y(0)E_{\gamma,1}(-\lambda x^\gamma) + y'(0)xE_{\gamma,2}(-\lambda x^\gamma) + \cdots + y^{(m-2)}x^{m-2}E_{\gamma,m-1}(-\lambda x^\gamma) + y^{(m-1)}x^{m-1}E_{\gamma,m}(-\lambda x^\gamma) \\
&= \sum_{k=1}^{m} y^{(k-1)}(0)x^{k-1}E_{\gamma,k}(-\lambda x^\alpha)
\end{aligned} \tag{31}
$$

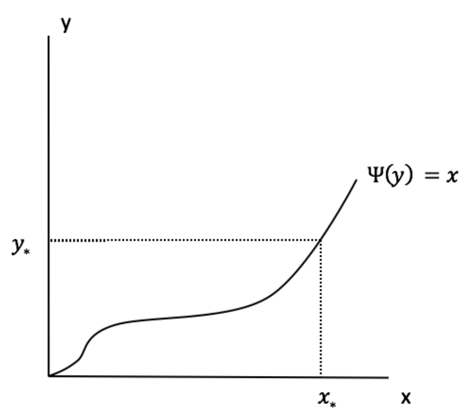
3 The Tautochrone Problem

The tautochrone problem asks the following question: *Determine a curve along which a heavy particle, sliding without friction, descends to its lowest point in a constant time, independent of its initial position*[2]. In 1673, Christiaan Huygens published *Horologium Oscillatorium*, in which he geometrically demonstrated that the solution to the tautochrone problem was a cycloid, but it wasn't until Niels Abel began working on the problem in 1823[6] that an analytical solution was found. Abel actually solved a more general version of the Tauochrone, and the result of this work can be recognized today as the definitions of the fractional integral and derivative[6]. In this section we will solve the standard tautochrone problem using the Laplace Transformation.

Before we begin we will define a mathematical object that will be useful soon: the *convolution*.
Definition 3.1: The *convolution* of two functions f and g, denoted $f * g$, is defined by

$$[f * g](x) = \int_0^x f(t)g(x-t)\,dt. \tag{32}$$

Now we may begin by considering an object of mass m falling under the force of gravity constrained to a curve given by $\Psi(y) = x$ seen in the figure below.



From the arc length formula, we know that

$$\frac{ds}{dy} = \sqrt{1 + \Psi'(y)^2}. \tag{33}$$

we also know that

$$-\frac{ds}{dt} = v. \tag{34}$$

with v representing the velocity of the particle. The time derivative of s is negative since the particle is falling as time increases. Since the curve we are examining is frictionless, we may apply conservation of energy. We will denote gravitational potential energy U and the kinetic energy T. Note that:

$$
\begin{aligned}
T(y) + U(y) &= U(y_*) \\
\frac{1}{2}mv^2 + mgy &= mgy_* \\
v &= \sqrt{2g(y_* - y)}.
\end{aligned}
$$

Note that $y < y_*$.We may now rearrange (34) to solve for time:

$$
dt = -\frac{ds}{v} \tag{35}
$$

$$
= -\frac{\sqrt{1 + \Psi'(y)^2}}{\sqrt{2g(y_* - y)}}dy. \tag{36}
$$

For the moment we will let $\phi(y) = \sqrt{(1 + \Psi'(y)^2)/2g}$, and integrating (36) from $t = 0$ to $t = t(y_*)$, and $y = y_*$ to $y = 0$ respectively we obtain:

$$
\begin{aligned}
\int_0^{t(y_*)} dt &= -\int_{y_*}^{0} \frac{\phi(y)}{\sqrt{y_* - y}}dy \\
&= \int_0^{y_*} \frac{\phi(y)}{\sqrt{y_* - y}}dy.
\end{aligned}
$$

Thus we arrive at a form of *Abel's Integral Equation of the first kind*:

$$
t(y_*) = \int_0^{y_*} \frac{\phi(y)}{\sqrt{y_* - y}}dy. \tag{37}
$$

Which is in fact the convolution of ϕ and $y^{-1/2}$, denoted as:

$$
t(y_*) = \phi * y_*^{-1/2}. \tag{38}
$$

The Laplace transform and convolutions have the very elegant property that the Laplace transform of a convolution is the same as the product of the Laplace transforms, i.e.

$$
\mathscr{L}\{g * h\}(s) = \mathscr{L}\{g\}\mathscr{L}\{h\}. \tag{39}
$$

We can apply this property to (38). Let $\mathscr{L}\{\phi\}(s) = \Phi(s)$, $\mathscr{L}\{t(y_*)\}(s) = T(s)$ and note that

$\mathscr{L}\{C\, t(x)\}(s)(s) = C\, T(s)$ for $C \in \mathbb{R}$. We also know that for $n \in \mathbb{Z}$:

$$\mathscr{L}\{y^n\}(s) = \frac{n!}{s^{n+1}}. \tag{40}$$

This may be generalized for $n \in \mathbb{R}$ be using the Gamma Function in place of $n!$. In this case (40) becomes

$$\mathscr{L}\{y^n\}(s) = \frac{\Gamma(n+1)}{s^{n+1}}. \tag{41}$$

Then it follows from (41) that the Laplace Transform of (38) is given by

$$T(s) = \Phi(s)\frac{\Gamma(1/2)}{s^{1+2}}. \tag{42}$$

Next we will isolate $\Phi(s)$ and take the inverse Laplace Transform of both sides:

$$\begin{aligned} \Phi(s) &= T(s)\frac{s^{1/2}}{\Gamma(1/2)} \\ \phi(y) &= \mathscr{L}^{-1}\Big\{T(s)\frac{s^{1/2}}{\Gamma(1/2)}\Big\}(y) \\ &= \mathscr{L}^{-1}\Big\{sT(s)\frac{s^{-1/2}}{\Gamma(1/2)}\Big\}(y). \end{aligned} \tag{43}$$

Let $h = \mathscr{L}^{-1}\{s^{-1/2}\}(y)$. Note that:

$$\begin{aligned} sF(y)\frac{s^{-1/2}}{\Gamma(1/2)} &= \frac{s}{\Gamma(1/2)}\mathscr{L}\{t(y_*)\}(s)\mathscr{L}\{h\}(s) \\ &= \frac{s}{\Gamma(1/2)}\mathscr{L}\{t * h\}(s) \\ &= \frac{1}{\Gamma(1/2)}\mathscr{L}\Big\{\frac{d}{dy_*}\big[t * h\big]\Big\}(s). \end{aligned} \tag{44}$$

To solve for h, observe the following:

$$\begin{aligned} \mathscr{L}^{-1}\{s^{-1/2}\}(y) &= \frac{1}{\Gamma(1/2)}\mathscr{L}^{-1}\Big\{\frac{\Gamma(1/2)}{s^{1/2}}\Big\}(y) \\ &= \frac{1}{\Gamma(1/2)}y^{-1/2}. \end{aligned} \tag{45}$$

Upon inserting (45) into (44) we obtain

$$\begin{aligned} sT(y)\frac{s^{-1/2}}{\Gamma(1/2)} &= \frac{1}{\Gamma(1/2)\Gamma(1/2)}\mathscr{L}\Big\{\frac{d}{dy_*}\big[t*y^{-1/2}\big]\Big\} \\ &= \frac{1}{\pi}\mathscr{L}\Big\{\frac{d}{dy_*}\big[t*y^{-1/2}\big]\Big\}. \end{aligned} \tag{46}$$

Finally we can insert this result into (43):

$$\begin{aligned} \phi(y) &= \mathscr{L}^{-1}\Big\{\frac{1}{\pi}\mathscr{L}\Big\{\frac{d}{dy_*}\big[t*y^{-1/2}\big]\Big\}\Big\}(y) \\ &= \frac{1}{\pi}\frac{d}{dy_*}\big[t*y^{-1/2}\big] \\ &= \frac{1}{\pi}\frac{d}{dy_*}\int_0^{y_*}\frac{t(y_*)}{(y_*-y)^{1/2}}\ dy. \end{aligned} \tag{47}$$

Equation (47) is the general solution to a form of Abel's Integral Equation of the first kind. However at the moment we would like to solve for the curve where $t(y_*) = k$ for some real number k. Then (47) becomes

$$\begin{aligned} \phi(y) &= \frac{k}{\pi}\frac{d}{dy_*}\int_0^{y_*}(y_*-y)^{-1/2}\ dy \\ &= \frac{2k}{\pi}\frac{d}{dy_*}\big(y_*^{1/2}\big) \\ &= \frac{k}{\pi}\frac{1}{\sqrt{y_*}}. \end{aligned}$$

Recall that we set $\phi(y) = \sqrt{1+\Psi'(y)^2/2g}$. Then it follows that

$$\begin{aligned} \sqrt{1+\Psi'(y)^2/2g} &= \frac{k}{\pi}\frac{1}{\sqrt{y_*}} \\ \sqrt{1+\Psi'(y)^2} &= \frac{k\sqrt{2g}}{\pi}\frac{1}{\sqrt{y_*}} \\ 1+\Psi'(y)^2 &= \frac{2gk^2}{\pi^2}\frac{1}{y_*} \\ &= 2R\frac{1}{y_*}. \end{aligned} \tag{48}$$

Here we let $R = \frac{gk^2}{\pi^2}$. A few more lines of algebra results in the following differential equation:

$$\Psi'(y) = \frac{dx}{dy_*} = \sqrt{\frac{2R}{y_*}-1}. \tag{49}$$

This is a separable ODE, so we will separate variables and integrate.

$$\int_0^x dt = \int_0^{y_*} \sqrt{\frac{2R}{y} - 1}\, dy. \tag{50}$$

Let $\sec^2(\theta) = 2R/y$. Then it follows that $dy = 4R\cos(\theta)\sin(\theta)\,d\theta$.
Substituting into (50) we find:

$$\begin{aligned}
x &= \int_0^\theta \sqrt{\sec^2(v) - 1}\; 4R\cos(v)\sin(v)\,dv \\
&= 4R\int_0^\theta \tan(v)\cos(v)\sin(t)\,dv \\
&= 4R\int_0^\theta \sin^2(v)\,dv \\
&= 2R\int_0^\theta 1 - \cos(2v)\,dv \\
&= 2R(\theta - \frac{1}{2}\sin(2\theta)) \\
&= R(2\theta - \sin(2\theta)). \qquad (51)
\end{aligned}$$

We have found that $x = R(2\theta - \sin(2\theta))$. We will return to this equation shortly, but first we will consider the expression we came across earlier:

$$dy = 4R\cos(\theta)\sin(\theta)\,d\theta. \tag{52}$$

Note that this is also a separable differential equation, and we will use it to solve for y.

$$\begin{aligned}
\int_0^y dt = y &= \int_0^\theta 4R\cos(v)\sin(v)\,dv \\
&= 2R(\sin^2(\theta)) \\
&= R(1 - \cos(2\theta)). \qquad (53)
\end{aligned}$$

If we let $2\theta = t$, where t is a real number, then equations (51) and (53) become :

$$x = R(t - \sin t) \tag{54}$$

$$y = R(1 - \cos t), \tag{55}$$

which is in fact the parametric equation of a cycloid.

4 The Tautochrone and Fractional Calculus

Recall the original integral equation (37) we obtained after setting up the tautochrone problem,

$$t(y_*) = \int_0^{y_*} \frac{\phi(y)}{\sqrt{y_* - y}}\, dy.$$

We will rewrite (37) as,

$$t(y_*) = \int_0^{y_*} \phi(y)\, (y_* - y)^{-1/2}\, dy. \tag{56}$$

Recall Riemann-Liouville Integral (1),

$$J^{\gamma} f = \frac{1}{\Gamma(\gamma)} \int_0^x f(t)(x-t)^{\gamma-1}\, dt.$$

Comparing (56) and (1), we note that they have the exact same structure. In fact, (56) is simply the Riemann-Liouville integral when $\gamma = 1/2$. Note that $\Gamma(1/2) = \sqrt{\pi}$. Then

$$t(y_*) = \sqrt{\pi}\, J^{1/2}\phi(y_*). \tag{57}$$

The truly amazing result comes when we observe the solution found in (47).

$$\begin{aligned} \phi(y_*) &= \frac{1}{\pi}\, D^1\, J^{1/2}\, t(y_*) \\ &= \frac{1}{\pi}\, D^{1/2}\, t(y_*). \end{aligned} \tag{58}$$

Equations (57) and (58) seem to imply that the fundamental theorem of calculus can be extended to the fractional realm. This was first observed by Abel in his 1823 paper on the Tautochrone[6]. However, for unknown reasons he did not study these ideas any further in subsequent papers. It is worth noting how incredible it is that an idea as abstract and initially unintuitive as fractional calculus naturally arises when solving a very tangible problem in physics.

5 Abel's Integral Equation of the Second Kind

There exists a further generalization of Abel's Equation given by:

$$\phi(x) - \frac{\lambda}{\Gamma(\alpha)} \int_0^x \frac{\phi(t)}{(x-t)^{1-\alpha}} dt = g(x). \tag{59}$$

We will use a method very similar to how we solved the tautochrone for this problem. We begin by rewriting (59) as follows:

$$\phi(x) - \frac{\lambda}{\Gamma(\alpha)} \big[\phi * x^{\alpha-1}\big] = g(x). \tag{60}$$

Now we take the Laplace Transform of both sides:

$$\begin{aligned} G(s) &= \Phi(s) - \frac{\lambda}{\Gamma(\alpha)} \mathscr{L}\{\phi * x^{\alpha-1}\}(s)(s) \\ &= \Phi(s) - \frac{\lambda}{\Gamma(\alpha)} \big[\Phi(s)\frac{\Gamma(\alpha)}{s^\alpha}\big] \\ &= \Phi(s)\big[1 - \frac{\lambda}{s^\alpha}\big]. \end{aligned} \tag{61}$$

Rearranging (61) we obtain:

$$\begin{aligned} \Phi(s) &= \frac{G(s)}{1 - \frac{\lambda}{s^\alpha}} \\ &= G(s)\frac{s^\alpha}{s^\alpha - \lambda} \\ &= s\big[G(s)\frac{s^{\alpha-1}}{s^\alpha - \lambda}\big]. \end{aligned} \tag{62}$$

We will now apply the inverse Laplace Transform.

$$\mathscr{L}^{-1}\{\Phi(s)\}(y) = \phi(x) = \mathscr{L}^{-1}\Big\{sG(s)\frac{s^{\alpha-1}}{s^\alpha - \lambda}\Big\}(y). \tag{63}$$

Let $h = \mathscr{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha-\lambda}\right\}$. Then

$$\begin{aligned}
\mathscr{L}^{-1}\Big\{sG(s)\frac{s^{\alpha-1}}{s^\alpha-\lambda}\Big\}(y) &= \mathscr{L}^{-1}\Big\{s\mathscr{L}\{g\}(s)\mathscr{L}\{h\}(s)\Big\}(y)\\
&= \mathscr{L}^{-1}\Big\{s\mathscr{L}\{g*h\}(s)\Big\}(y)\\
&= \mathscr{L}^{-1}\Big\{\mathscr{L}\Big\{\frac{d}{dx}(g*h)\Big\}\Big\}(y)\\
&= \frac{d}{dx}(g*h). \qquad (64)
\end{aligned}$$

Now we want to figure out what h is. Note the following:

$$\begin{aligned}
\frac{s^{\alpha-1}}{s^\alpha-\lambda} &= s^{-1}\frac{s^\alpha}{s^\alpha-\lambda}\\
&= s^{-1}\frac{1}{1-\lambda s^{-\alpha}}\\
&= s^{-1}\sum_{k=0}^{\infty}(\lambda s^{-\alpha})^k\\
&= \sum_{k=0}^{\infty}\lambda^k s^{-\alpha k-1}\\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{s^{\alpha k+1}}\\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+1)}\frac{\Gamma(\alpha k+1)}{s^{\alpha k+1}}. \qquad (65)
\end{aligned}$$

Taking the inverse Laplace of (65) we find

$$\begin{aligned}
\mathscr{L}\Big\{\sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+1)}\frac{\Gamma(\alpha k+1)}{s^{\alpha k+1}}\Big\} &= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+1)}\mathscr{L}\Big\{\frac{\Gamma(\alpha k+1)}{s^{\alpha k+1}}\Big\}\\
&= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha k+1)}x^{\alpha k}\\
&= \sum_{k=0}^{\infty}\frac{(\lambda x^\alpha)^k}{\Gamma(\alpha k+1)}\\
&= E_\alpha(\lambda x^\alpha). \qquad (66)
\end{aligned}$$

Therefore $h = E_\alpha(\lambda x^\alpha)$, and the solution to Abel's Integral Equation of the second kind is given by

$$\phi(x) = \frac{d}{dx}\big[g * E_\alpha(\lambda x^\alpha)\big]. \tag{67}$$

We may examine some of the curves that arise from various values of the g term. Consider the case when $g = x^\beta$, with $\beta \in \mathbb{R}^+$. Substituting $g = x^\beta$ into (67) we find that,

$$\begin{aligned}\phi(x) &= \frac{d}{dx}\big[x^\beta * E_\alpha(\lambda x^\alpha)\big] \\ &= \frac{d}{dx}\int_0^x t^\beta \sum_{j=0}^{\infty} \frac{\lambda^j (x-t)^{j\alpha}}{\Gamma(j\alpha+1)}\,dt \\ &= \frac{d}{dx}\sum_{j=0}^{\infty}\frac{\lambda^j}{\Gamma(j\alpha+1)}\int_0^x t^\beta (x-t)^{j\alpha}\,dt. \end{aligned} \tag{68}$$

We once again come across the Beta function. In this case, the solution to the integral in (68) is,

$$\int_0^x t^\beta (x-t)^{j\alpha}\,dt = \frac{\Gamma(\beta+1)\Gamma(j\alpha+1)}{\Gamma(\beta+j\alpha+2)}(x-t)^{j\alpha+\beta+1}$$

Upon substituting into (68),

$$\begin{aligned}\phi(x) &= \frac{d}{dx}\sum_{j=0}^{\infty}\frac{\lambda^j}{\Gamma(j\alpha+1)}\frac{\Gamma(\beta+1)\Gamma(j\alpha+1)}{\Gamma(\beta+j\alpha+2)}(x-t)^{j\alpha+\beta+1} \\ &= \frac{d}{dx}\sum_{j=0}^{\infty}\frac{\lambda^j\Gamma(\beta+1)}{\Gamma(j\alpha+\beta+2)}x^{j\alpha+\beta+1} \\ &= \sum_{j=0}^{\infty}\frac{\lambda^j\Gamma(\beta+1)}{\Gamma(j\alpha+\beta+2)}\frac{d}{dx}\big[x^{j\alpha+\beta+1}\big] \\ &= \sum_{j=0}^{\infty}\frac{\lambda^j\Gamma(\beta+1)}{\Gamma(j\alpha+\beta+2)}(j\alpha+\beta+1)x^{j\alpha+\beta} \\ &= x^\beta\Gamma(\beta+1)\sum_{j=0}^{\infty}\frac{(\lambda\, x^\alpha)^j}{\Gamma(j\alpha+\beta+1)} \\ &= x^\beta\Gamma(\beta+1)E_{\alpha,\beta+1}(\lambda x^\alpha).\end{aligned}$$

6 Ongoing Investigation

When we solve the tautochrone problem, we considered a curve where $t(y_*)$ is a constant. A natural question to ask next is if there are any curves that come of equation (47) when time is not constant. Muñoz and Fernández-Anaya in [7] provide an excellent analysis of this consideration for $\Delta t \propto (\Delta y)^\beta$. The primary result from this paper is the discovery that there is no solution for $-1/2 < \beta \leq 1/2$.

We are currently exploring the following extension,

$$\Delta t \propto M(y_*) \tag{69}$$

Where $M(y_*)$ is a real valued function of the initial position y_*. The following equations are what we found when we substituted various M functions into (47),

$$\frac{dx}{dy} = \frac{1}{\sqrt{\pi y_*}} E_{1,1/2}(\lambda y_*) \qquad \text{for } \Delta t \propto e^{\lambda y_*} \tag{70}$$

$$\frac{dx}{dy} = \frac{\lambda\sqrt{y_*}}{\sqrt{\pi}} E_{2,3/2}(-(\lambda y_*)^2) \qquad \text{for } \Delta t \propto \sin(\lambda y_*) \tag{71}$$

$$\frac{dx}{dy} = \frac{1}{\sqrt{\pi y_*}} E_{2,1/2}((\lambda y_*)^2) \qquad \text{for } \Delta t \propto \cos(\lambda y_*) \tag{72}$$

$$\frac{dx}{dy} = \frac{1}{\sqrt{\pi y_*}} \sum_{j=1}^{\infty} \frac{\Gamma(j)(-1)^j}{\Gamma(j+1/2)} y_*^j \qquad \text{for } \Delta t \propto \ln(y_* + 1) \tag{73}$$

Recall that when we set t proportional to a constant, we found that

$$\frac{dx}{dy} = \frac{k}{\pi} \frac{1}{\sqrt{y_*}}$$

Note that in (70), (72), and (73) we find that the tautochrone pops up as a coefficient. It is particularly interesting that $\sin(\lambda y_*)$ does not have this tautochrone coefficient.

We are also investigating the possibility of using the results from [7] to develop a possible test to see what kind of curves will yield a legitimate solution. That is, real valued curves. This would be much better than attempting to directly integrate the curves in equations (70), (72), and (73). Since the boundaries on β from [7] are quite small, it looks like valid curves only come out of (47) when Δt is reasonably close to zero. It would follow from this that if $\Delta t < \Delta y^{1/2}$, then a valid result could be obtained.

We have found the following corollary from Muñoz's and Fernández-Anaya's result.

Corollary. *If time is proportional to a real, differentiable function f such that,*

$$f^{(k)}(y_0) > 0,$$

for all $k \in \mathbb{N}$, then no curve exists satisfying the conditions of Abel's Integral Equation. That is, a particle falling under the influence of gravity cannot follow a path in which time is proportional to f. Furthermore, we recover the tautochrone if $f^{(k)}(y_0) = 0$ for all $k > 0$.

Proof. Let $\Delta t \propto f$, where f is a real, differentiable function on the interval (y, y_*) and $f^{(k)}(y_0) > 0$ for $k > 0$ and $y_0 \in (y, y_*)$. It follows that,

$$\begin{aligned} f &= f(y_0) + f'(y_0)(y - y_0) + \frac{f''(y_0)}{2}(y - y_0)^2 + \cdots + \frac{f^{(k)}(y_0)}{\Gamma(k+1)}(y - y_0)^k \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(y_0)}{\Gamma(k+1)}(y - y_0)^k. \end{aligned}$$

Now if we insert f into (47),

$$\begin{aligned} \phi(y) &= \frac{k}{\pi}\frac{d}{dy_*}\int_0^{y_*} f(y)(y_* - y)^{-1/2}\,dy \\ &= \frac{k}{\pi}\frac{d}{dy_*}\int_0^{y_*} \sum_{k=0}^{\infty} \frac{f^{(k)}(y_0)}{\Gamma(k+1)}(y - y_0)^k (y_* - y)^{-1/2}\,dy \\ &= \frac{k}{\pi}\frac{d}{dy_*}\sum_{k=0}^{\infty} \frac{f^{(k)}(y_0)}{\Gamma(k+1)}\int_0^{y_*} (y - y_0)^k (y_* - y_0)^{-1/2}\,dy. \end{aligned} \tag{74}$$

The integral in (74) can be analyzed using the results of Muñoz and Fernández-Anaya. We know that $(y - y_0)^k$ will only be a valid solution if $-1/2 < k \leq 1/2$. However, $k \geq 0$, therefore after $k = 0$ no terms in the series expansion of f will yield a real solution for arbitrarily small values of $(y - y_0)$. Consequently, a solution exists only if $f^{(k)}(y_0) = 0$ for all $k > 0$. This case leads to the tautochrone curve since f would be constant.

□

We are currently looking into the case when the Taylor series of a given function f is alternating.

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