Derivative Security Pricing with the Binomial Asset Pricing Model

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December 13, 2019

Milledgeville, Georgia

Abstract

A derivative security serves as a financial agreement in which a buyer is given the right to purchase assets at a predetermined price, which can be exercised at any point before the expiration of the security. In this paper, we explore the Binomial Asset Pricing Model and how it is used to price derivative securities for both stocks and bonds in the risk-neutral world. We look at how call options are regressively priced under risk-neutral probabilities that result in stock profit growth equal to that of the money market returns. Similarly, we examine interest rate discount processes and their effects on zero-coupon bonds prices, bond wealth portfolios, and derivative securities.

Acknowledgments

I would like to thank Dr. Jebessa Mijena for his support, encouragement, and advisement throughout the course of my research. Additionally, a thank you to Dr. Brent Evans who has advanced my understanding of many of the finance related topics in this paper. Lastly, I am grateful to the faculty of the Mathematics Department at Georgia College and State University for developing my mathematical abilities and provided ample opportunities for academic growth throughout my undergraduate career.

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1 Introduction to the Binomial Asset Pricing Model

1.1 Binomial Models

The model discussed in this paper follows a discrete binomial distribution, that is,

Definition 1. A random variable X has a **binomial distribution with parameters** n **and** p if X has a discrete distribution for which the p.f. is as follows:

$$f(x|n,p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{for } x = 0, 1, 2, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

where $n \in \mathbb{Z}^+$ and $0 \leq p \leq 1$.

Additionally, the mean, variance, and standard deviation of a random variable X with a binomial distribution with parameters n and p, respectively, are as follows:

$$\begin{aligned} E(X) &= \sum_{i=1}^{n} E(X_i) = np, \\ Var(X) &= \sum_{i=1}^{n} Var(X_i) = np(1-p), \\ Std(X) &= \left[\sum_{i=1}^{n} Var(X_i) = np(1-p) \right]^{\frac{1}{2}} = \sqrt{np(1-p)}. \end{aligned}$$

In the case of the Binomial Asset Pricing Model, the two mutually exclusive outcomes of a trial are given by the upstate, meaning that a given variable rises in value, or the downstate, if the variable has dropped in value. We let u be the factor by which an asset rises when moving to the up-state, and d be the factor by which an asset changes when moving to the down-state. We also refer to each trial as a period, and assume that these periods are independent of each other.

1.2 Arbitrage

In our model, we assume there is no arbitrage. That is, there is no possibility to begin with zero wealth, have a zero-probability of losing money, and positive probability of earning money through investments. in order to rule out arbitrage, we assume the following inequality:

$$0 < d < 1 + r < u,$$

where d is the down-factor, u is the up-factor, and r is the fixed interest rate of the money market. The inequality $0 < d$ ensures that the value of a stock price, even in its downstate, remains positive. The second term of our inequality, $d < 1 + r$, focuses on ruling out arbitrage. If we instead assume $d \geq 1+r$, then there is a possibility to begin with an no initial wealth, borrow an amount x to invest in assets, and even in the event of a failure our downstate price xd is larger than the amount owed to the bank $x(1 + r)$. Thus, we have zero probability of losing money because we keep a profit of $xd - x(1 + r)$ after repaying the loan including interest. Therefore, $d \geq 1 + r$ allows for arbitrage, and we must assume $d < 1 + r$. Finally, $1 + r < u$ gives incentive for asset investment. If this were not the case, that is $1 + r \geq u$, then the money market would yield greater returns with less risk, which would disincentives agents to invest in assets.

1.3 Derivative Securities

A derivative security serves as a financial agreement that has value dependent upon some agreed upon asset. The payoff and price of a derivative security depend specifically on the type of security, and differs for the call options, zero-coupon bonds, and fixed income derivatives that we examine in this paper. In general, we define the price of a derivative, V_0, recursively based on the expected payoff, V_t, of the security where t is a future time period.

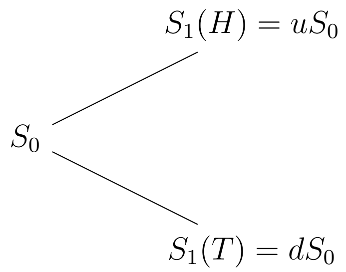
1.4 Risk Neutral Probabilities

As will be seen throughout this paper, pricing derivative securities requires the use of risk-neutral measures. Risk neutral probability measures are similarly to traditional probabilities measures, however the probabilities themselves are adjusted for the risk that are taken on in purchasing assets. By taking risk into account, we are able to price derivative securities with consideration to a buyers risk aversion. The use of these measures gives rise to many convenient properties that will be explores later in this paper, and aid in pricing derivative for stock options and bond securities.

2 Call Options in the Binomial Asset Pricing Model

2.1 Single Period Model

A One-Period Binomial Asset Pricing Model for stock investment begins with an initial stock price, S_0, and takes on the value of either the upstate or downstate depending on the outcome of a trial. We represent this trial with a coin toss, where heads results in a price rising to upstate and tails results in a price falling to downstate. The outcome of the coin toss is random and each possible outcome is mutually exclusive as discussed in Section 1. If the toss is heads, the price of the upstate, $S_1(H)$, is the product of the original price and the up-factor. That is, $S_1(H) = uS_0$. Similarly, the downstate price in the case of a coin toss of tails is $S_1(T) = dS_0$. In general, the one-period binomial model is given below.



The derivative security, or call option, can be bought for a price V_0 and allows the buyer to purchase stock for an agreed upon strike price, K. At any time t before the expiration of the option, this price may be exercised for a payoff V_t. In particular, for a single period model the payoff of the option exercised at time 1 is given by

$$V_1 = (S_1 - K)^+$$

denoting that the payoff value is the maximum of $S_1 - K$ and 0. In the case that $S_1 > K$, the option is *in the money*, and thus the option can be exercised for profit at time $t = 1$. Alternatively, when $S_1 < K$ the option is *out of the money* at time $t = 1$ and $V_1 = 0$. Thus the option can not be immediately exercised for profit. However, it may gain value in future periods if the stock price continues to rise. In the case that $S_1 = K$, the option is *at the money*.

In this model, an agents initial wealth, X_0 is held in either stocks, the money market, or a combination of both between 0 to time 1. If a buyer invest in Δ_0 shares of stock at time 0 for a stock price S_0, then there is a total of $\Delta_0 S_0$ dollars invested in stocks. Additionally, those shares take on a price of S_1 in time 1 yielding a profit of $\Delta_0 S_1$ from these stock investments in time 1. Additionally, the remaining amount of initial wealth that is not invested in stocks, $X_0 - \Delta_0 S_0$ is invested into the money market and makes a return of $(1+r)$. Therefore, the wealth of the agent at time one is given by the following equation:

$$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0).$$

By distributing the money market returns and factoring Δ_0, we have that

$$X_1 = (1+r)X_0 + \Delta_0(S_1 - (1+r)S_0).$$

Our goal in pricing an option is to choose an initial stock investment and wealth, Δ_0 and X_0, such that $X_1(H) = V_1(H)$ and $X_1(T) = V_1(T)$ in order to price our call option. Recall

$$X_1 = (1+r)X_0 + \Delta_0(S_1 - (1+r)S_0).$$

Thus for $V_1 = X_1$ we must have that

$$V_1 = (1+r)X_0 + \Delta_0(S_1 - (1+r)S_0).$$

Dividing by our money market returns yields that

$$\tfrac{1}{1+r}V_1 = X_0 + \Delta_0\Big(\tfrac{1}{1+r}S_1 - S_0\Big).$$

In particular, for each outcome we have

$$\tfrac{1}{1+r}V_1(H) = X_0 + \Delta_0\Big(\tfrac{1}{1+r}S_1(H) - S_0\Big),$$
$$\tfrac{1}{1+r}V_1(T) = X_0 + \Delta_0\Big(\tfrac{1}{1+r}S_1(T) - S_0\Big).$$

We let $\tilde{p}$ be the risk neutral probability that the toss is heads, and $\tilde{q}$ such that $\tilde{q} = 1 - \tilde{p}$ be the risk neutral probability that the toss is tails. By multiplying $\tilde{p}$ and $\tilde{q}$ to each respective equation above, and adding these together, we have that

$$\frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)] = X_0 + \Delta_0\Big(\frac{1}{1+r}[\tilde{p}S_1(H) + \tilde{q}S_1(T)] - S_0\Big).$$

We wish to define these risk neutral probabilities such that the current stock price is equivalent to the expected value of the future stock price discounted for the interest accrued. That is,

$$S_0 \quad = \quad \frac{1}{1+r}[\tilde{p}S_1(H) + (1-\tilde{p})S_1(T)].$$

By substituting for the upstate and down state stock price, it follows that

$$S_0 = \frac{1}{1+r}[\tilde{p}uS_0 + (1-\tilde{p})dS_0].$$

Which implies

$$\frac{1}{1+r}[\tilde{p}u + \tilde{q}d] = 1.$$

This can be used to solve for our risk nuetral probabilities. Therefore, $\tilde{p}$ and $\tilde{q}$ are defined as follows:

$$\tilde{p} = \tfrac{1+r-d}{u-d} \text{ and } \tilde{q} = \tfrac{u-1-r}{u-d}.$$

Our choice of $\tilde{p}$ and $\tilde{q}$ results in stock prices such that $\Big(\tfrac{1}{1+r}[\tilde{p}S_1(H)+\tilde{q}S_1(T)]-S_0\Big) = 0$. Therefore, equation from above,

$$\frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)] = X_0 + \Delta_0\Big(\frac{1}{1+r}[\tilde{p}S_1(H) + \tilde{q}S_1(T)] - S_0\Big),$$

can now be written as

$$X_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)].$$

Given X_0, we chose to buy Δ_0 shares of stock. Then the portfolio is worth either $V_1(H)$ or $V_1(T)$. The result is that the derivative security should be priced at

$$V_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)].$$

2.2 Multiperiod Model

In order to extend our single period model into multiple finite periods, we will begin by generalizing our three main equations found in the previous section. The wealth equation for a general time n is given by

$$X_n = \Delta_{n-1}S_n + (1+r)(X_{n-1} - \Delta_{n-1}S_{n-1}).$$

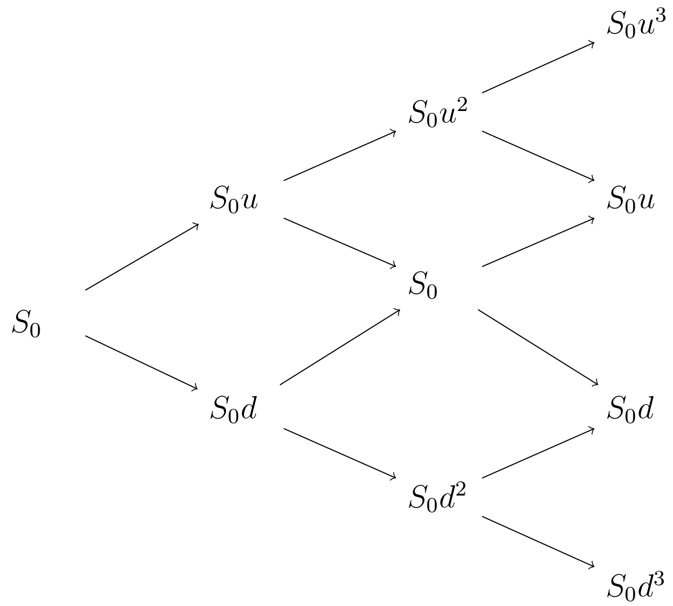
Additionally, the payoff of a call option at time n is given by

$$V_n = (S_n - K)^+$$

Finally, we can define the price of a call option at time n based on the expected payoff of the option in the time $n+1$ as follows:

$$V_n(\omega_1\omega_2\dots\omega_n) = \tfrac{1}{1+r}[\widetilde{p}V_{n+1}(\omega_1\omega_2\dots\omega_n H) + \widetilde{q}V_{n+1}(\omega_1\omega_2\dots\omega_n T)]$$

Where the sequence $\omega_1\omega_2\dots\omega_n$ denotes the result of the first n coin tosses and $\tilde{p}$ and $\tilde{q}$ are defined the same as in the previous section. In order to visualize the possible outcomes of the model, we can extend the single-period tree model in Section 2.1 as seen below.



Because our model follows a binomial distribution, we have that the stock price can take 2^n possible paths throughout the model, where n is the number of periods.

Theorem 1 *Consider an $N-period$ binomial asset pricing model with no arbitrage and with*

$$\tilde{p} = \tfrac{1+r-d}{u-d},\ \tilde{q} = \tfrac{u-1-r}{u-d}.$$

Let V_n be a random variable derivative security paying off at time N which depends on the first N coin tosses $\omega_1\omega_2\dots\omega_N$. Define sequence of random variables $V_{N-1}, V_{N-2}\dots V_0$ recursively by the equation

$$V_n(\omega_1\omega_2\dots\omega_n) = \tfrac{1}{1+r}[\widetilde{p}V_{n+1}(\omega_1\omega_2\dots\omega_n H) + \widetilde{q}V_{n+1}(\omega_1\omega_2\dots\omega_n T)],$$

so that each V_n depends on the first n coin tosses $\omega_1\omega_2\dots\omega_n$, where n falls in the interval $[0, N-1]$. Next, define the delta-hedging formula to be

$$\Delta_n(\omega_1\omega_2\ldots\omega_n) = \frac{V_{n+1}(\omega_1\omega_2\ldots\omega_n H) - V_{n+1}(\omega_1\omega_2\ldots\omega_n T)}{S_{n+1}(\omega_1\omega_2\ldots\omega_n H) - S_{n+1}(\omega_1\omega_2\ldots\omega_n T)}.$$

If we set $X_0 = V_0$ and define the portfolio values $X_1, X_2, \ldots X_N$ by the wealth equation

$$X_n = \Delta_{n-1}S_n + (1+r)(X_{n-1} - \Delta_{n-1}S_{n-1})$$

then we will have that

$$X_N(\omega_1\omega_2\ldots\omega_N) = V_N(\omega_1\omega_2\ldots\omega_N)$$

for all sequences $\omega_1\omega_2\ldots\omega_N$

Proof. We will prove Theorem 1 with proof by induction. Let there be N periods and no arbitrage. Define $\tilde{p}, \tilde{q}$, X_n,, Δ_n, and V_n as follows:

$$\begin{aligned}
\tilde{p} &= \frac{1+r-d}{u-d}, \\
\tilde{q} &= \frac{u-1-r}{u-d}, \\
X_n &= \Delta_{n-1}S_n + (1+r)(X_{n-1} - \Delta_{n-1}S_{n-1}) \\
V_n(\omega_1\omega_2\ldots\omega_n) &= \frac{1}{1+r}[\widetilde{p}V_{n+1}(\omega_1\omega_2\ldots\omega_n H) + \widetilde{q}V_{n+1}(\omega_1\omega_2\ldots\omega_n T)], \\
\Delta_n(\omega_1\omega_2\ldots\omega_n) &= \frac{V_{n+1}(\omega_1\omega_2\ldots\omega_n H) - V_{n+1}(\omega_1\omega_2\ldots\omega_n T)}{S_{n+1}(\omega_1\omega_2\ldots\omega_n H) - S_{n+1}(\omega_1\omega_2\ldots\omega_n T)}.
\end{aligned}$$

Assume $V_0 = X_0$, and we will show $X_N(\omega_1\omega_2\ldots\omega_N) = V_N(\omega_1\omega_2\ldots\omega_N)$ for all sequences $\omega_1\omega_2\ldots\omega_N$. For the base case, $n = 0$, we have that $V_0 = X_0$ by hypothesis. Now let $\omega_1\omega_2\ldots\omega_n\omega_{n+1}$ be a fixes arbitrary sequence and assume $X_n(\omega_1\omega_2\ldots\omega_n) = V_n(\omega_1\omega_2\ldots\omega_n)$ for $n < N$. We must consider two cases: $\omega_{n+1} = H$ and $\omega_{n+1} = T$. We will shorten our notation to have that $X_{n+1}(\omega_1\omega_2\ldots\omega_n H) = X_{n+1}(H)$, $V_{n+1}(\omega_1\omega_2\ldots\omega_n H) = V_{n+1}(H)$, $X_{n+1}(\omega_1\omega_2\ldots\omega_n T) = X_{n+1}(T)$, and $V_{n+1}(\omega_1\omega_2\ldots\omega_n T) = V_{n+1}(T)$

Case 1. Suppose $\omega_{n+1} = H$. By our wealth equation, we have that

$$\begin{aligned}
X_{n+1}(H) &= \Delta_n u S_n + (1+r)(X_n - \Delta_n S_n) \\
&= \left[\frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)}\right] uS_n + (1+r)(X_n - \Delta_n S_n) \\
&= \left[\frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n}\right] uS_n + (1+r)(X_n - \Delta_n S_n) \\
&= (1+r)X_n + \Delta_n S_n(u - (1_r)) \\
&= (1+r)V_n + \frac{V_{n+1}(H) - V_{n+1}(T))(u-(1+r))}{u-d} \\
&= (1+r)V_n + \tilde{q}V_{n+1}(H) - \tilde{q}V_{n+1}(T) \\
&= \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) + \tilde{q}V_{n+1}(H) - \tilde{q}V_{n+1}(T) \\
&= \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(H) \\
&= V_{n+1}(H).
\end{aligned}$$

Thus, $X_{n+1}(H) = V_{n+1}(H)$. That is, $X_{n+1}(\omega_1\omega_2\ldots\omega_n H) = V_{n+1}(\omega_1\omega_2\ldots\omega_n H)$ for any sequence $\omega_1\omega_2\ldots\omega_n$.

Case 2. Let $\omega_{n+1} = T$. Without loss of generality, we have that $X_{n+1}(\omega_1\omega_2 \dots \omega_n T) = V_{n+1}(\omega_1\omega_2 \dots \omega_n T)$ for any sequence $\omega_1\omega_2 \dots \omega_n$.

Consequently, $X_{n+1}(\omega_1\omega_2 \dots \omega_n\omega_{n+1}) = V_{n+1}(\omega_1\omega_2 \dots \omega_n\omega_{n+1})$. ■

2.3 AAPL Option Pricing

We will look at an example of pricing call options with the multiperiod model using Apple Inc. stock data from a one year span (Sept. 26, 2018 to Sept. 26, 2019). Let money market returns be a constant 2% and assume the model has four periods. In choosing an up-factor, we found the average rise in price from the beginning of a given quarter to its end, given the price rose in that quarter. Additionally, the down-factor was calculated by taking the reciprocal of the up-factor. Finally, we used the risk neutral probability formulas which we previously defined to calculate $\tilde{p}$ and $\tilde{q}$. The results are listed in the table below.

Time Period	r	u	d	p	q
0.25	0.02	1.1196	0.8932	0.5602	0.4398

In order to visualize the possible prices the stock takes on, we create a binomial tree model which begins with the original stock price of our data, $\$220.42$. From there, we calculate each upstate price to be the product of the previous stock price and the up-factor. Similarly, each downstate price is found to be the product of the price and the down-factor. Therefore, we have the following:

t=0	t=1	t=2	t=3	t=4
				\$346.28
			\$309.30	
		\$276.27		\$276.27
	\$246.77		\$246.77	
\$220.42		\$220.42		\$220.42
	\$196.88		\$196.88	
		\$175.86		\$175.86
			\$157.08	
				\$140.30

Example 1. Take $K = \$220.42$ and note that the option is at the money. It follows that the payoff for each of the possible stock priced in period 4 ordered from highest to lowest are as follows:

$$V_4(HHHH) = (346.28 - 220.42)^+ = 125.86,$$

$$V_4(HHHT, HHTH, HTHH, THHH) = (276.27 - 220.42)^+ = 55.85,$$

$$V_4(TTHH, THTH, THHT, HTTH, HTHT, HHTT) = (220.42 - 220.42)^+ = 0,$$

$$V_4(HTTT, THTT, TTHT, TTTH) = (175.86 - 220.42)^+ = 0,$$

$$V_4(TTTT) = (140.30 - 220.42)^+ = 0$$

Next, we use the formula $V_n(\omega_1\omega_2 \dots \omega_n) = \frac{1}{1+r}[\tilde{p}V_{n+1}(\omega_1\omega_2 \dots \omega_n H) + \tilde{q}V_{n+1}(\omega_1\omega_2 \dots \omega_n T)]$ to recursively price each option, until we are left with a single time 0 call option price. For example,

$$V_3(HHH) = \frac{1}{1+r}[\tilde{p}V_4(HHHH) + \tilde{q}V_4(HHHT)] = 93.21$$

$V_3(HHT, HTH, THH) = \frac{1}{1+r}[\tilde{p}V_4(HHTH) + \tilde{q}V_4(HHTT)] = 30.67$

$V_3(TTH, THT, HTT) = \frac{1}{1+r}[\tilde{p}V_4(TTHH) + \tilde{q}V_4(TTHT)] = 0$

$V_3(TTTT) = \frac{1}{1+r}[\tilde{p}V_4(TTTH) + \tilde{q}(TTTT] = 0$

Next, for period two we have that

$V_2(HH) = \frac{1}{1+r}[\tilde{p}V_3(HHH) + \tilde{q}(HHT] = 64.42$

$V_2(HT, TH) = \frac{1}{1+r}[\tilde{p}V_3(HTH) + \tilde{q}(HTT] = 16.85$

$V_2(TT) = \frac{1}{1+r}[\tilde{p}V_3(TTH) + \tilde{q}(TTT] = 0$

And thus

$V_1(H) = \frac{1}{1+r}[\tilde{p}V_2(HH) + \tilde{q}V_2(HT)] = 42.64$

$V_1(T) = \frac{1}{1+r}[\tilde{p}V_2(TH) + \tilde{q}V_2(TT)] = 9.25$

Finally,

$V_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)] = 27.41.$

We organize the potential prices of the option at each period in the following table:

V4	V3	V2	V1	V0
$125.86				
	$93.21			
$55.85		$64.41		
	$30.67		$42.64	
$0.00		$16.85		$27.41
	$0.00		$9.25	
$0.00		$0.00		
	$0.00			
$0.00				

Thus, a call option spanning our data set with a strike price of $\$220.42$ would be priced at $\$27.42$.

Example 2. Now let $K = 210.50$. We can again regressively price the option as before, resulting in the following tree:

V4	V3	V2	V1	V0
$135.78				
	$102.93			
$65.77		$73.95		
	$40.40		$51.19	
$9.92		$24.54		$34.48
	$5.45		$14.76	
$0.00		$2.99		
	$0.00			
$0.00				

Thus, we conclude that our option cost $\$34.48$. Note that this in the money option is more expensive than the previous example which is at the money. Because this example is in the money, it could be exercised immediately for profit, and thus one must pay more for this option than one which can not yield immediate returns.

Example 3. Now let $K = 231.83$. We can again regressively price the option as before, resulting in the following tree:

V4	V3	V2	V1	V0
$114.45				
	$82.02			
$44.44		$55.57		
	$24.41		$36.30	
$0.00		$13.40		$23.11
	$0.00		$7.36	
$0.00		$0.00		
	$0.00			
$0.00				

Thus, we conclude that our option cost $\$23.11$. Note that this option is less expensive than both previous examples. Because this example is out of the money, the stock price must rise further to exceed the strike price and be exercised for profit. Therefore, one pays less for this option which has lower likelihood of being exercised for profit than a option which is in or at the money.

3 Properties of Risk Neutral Probability

In this section, we will examine certain properties that hold under risk neutral probability measures. Take Ω to be a finite sample space such that

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\},$$

and let $\tilde{p}$ be the risk neutral probability of heads and $\tilde{q}$ be the risk neutral probability of tails under risk the risk neutral probability measure $\widetilde{\mathbb{P}}$. The following properties hold:

i. $\sum\limits_{\omega \in \Omega} \widetilde{\mathbb{P}}(\omega) = 1$

ii. $\widetilde{\mathbb{P}}(HHH) = \widetilde{p}^3$; $\widetilde{\mathbb{P}}(HHT) = \widetilde{p}^2\widetilde{q}$; $\widetilde{\mathbb{P}}(HTH) = \widetilde{p}^2\widetilde{q}$; $\widetilde{\mathbb{P}}(THH) = \widetilde{p}^2\widetilde{q}$;
$\widetilde{\mathbb{P}}(HTT) = \widetilde{p}\widetilde{q}^2$; $\widetilde{\mathbb{P}}(THT) = \widetilde{p}\widetilde{q}^2$; $\widetilde{\mathbb{P}}(TTH) = \widetilde{p}\widetilde{q}^2$ and $\widetilde{\mathbb{P}}(TTT) = \widetilde{q}^3$

iii. $\widetilde{\mathbb{E}}X = \sum\limits_{\omega \in \Omega} X(\omega)\widetilde{\mathbb{P}}(\omega)$, where X is a random variable defined on $(\Omega, \widetilde{\mathbb{P}})$.

3.1 Conditional Expectations

In the model we have defined, the value of a variable at time $n+1$ is dependent on not only ω_{n+1}, but also the sequence of tosses $\omega_1\omega_2 \ldots \omega_n$. Thus, each variable has a conditional probability distribution. In order to calculate the expected value of a conditional variable, we must formulate a method of finding conditional expectations.

Definition 2. Let n satisfy $1 \leq n \leq N$, and let $\omega_1\omega_2 \ldots \omega_n$ be given and fixed. There are 2^{N-n} possible continuations $\omega_{n+1} \ldots \omega_N$ of the sequence fixed $\omega_1\omega_2 \ldots \omega_n$. Denote the number of heads in the sequence of continuation $\omega_{n+1} \ldots \omega_N$ by $\#H(\omega_{n+1} \ldots \omega_N)$ and the number of tails as $\#T(\omega_{n+1} \ldots \omega_N)$. We define

$$\widetilde{\mathbb{E}}_n[X](\omega_1\omega_2 \ldots \omega_n) = \textstyle\sum_{\omega_{n+1} \ldots \omega_N} \tilde{p}^{\#H()(\omega_{n+1} \ldots \omega_N)} \tilde{q}^{\#T((\omega_{n+1} \ldots \omega_N))} X(\omega_1 \ldots \omega_n \omega_{n+1 \omega_N})$$

and call $\widetilde{\mathbb{E}}_n[X]$ the **conditional expectation of X based on the information at time n.** The two extreme cases of conditioning are $\widetilde{\mathbb{E}}_0[X]$, the conditional expectation of X based on no information, which we define by

$$\widetilde{\mathbb{E}}_0[X] = \widetilde{\mathbb{E}}X,$$

and $\widetilde{\mathbb{E}}_N[X]$, the conditional expectation of X based on knowledge of all N coin tosses, which we define by

$$\widetilde{\mathbb{E}}_N[X] = X.$$

We will examine the following properties of conditional expectations under risk neutral probabilities.

Theorem 2 *Let $N \in \mathbb{Z}^+$, and let X and Y be random variables depending on $\omega_1\omega_2 \ldots \omega_N$. Assume $0 \leq n \leq N$. The following properties hold true:*

1. ***Linearity of conditional expectations:*** *Let c_1 and c_2 be constants. We have that*

$$\widetilde{\mathbb{E}}_n[c_1X + c_2Y] = c_1\widetilde{\mathbb{E}}_n[X] + c_2\widetilde{\mathbb{E}}_n[Y].$$

2. ***Taking out whats is known:*** *If X only depends on the first n tosses, then*

$$\widetilde{\mathbb{E}}_n[XY] = X \cdot \widetilde{\mathbb{E}}_n[Y].$$

3. ***Iterated conditioning:*** *If 0 then*

$$\widetilde{\mathbb{E}}_n\left[\widetilde{\mathbb{E}}_m[X]\right] = \widetilde{\mathbb{E}}_n[X].$$

4. ***Independence:*** *If X depends only on tosses greater than or equal to $n+1$, then*

$$\widetilde{\mathbb{E}}_n[X] = \widetilde{\mathbb{E}}X.$$

Proof. We will prove each of the four properties of conditional expectations below.

Property 1. Observe the following:

$$\begin{aligned}
\widetilde{\mathbb{E}}_n[c_1X + c_2Y] &= \sum_{\omega_{n+1}\ldots\omega_N} \tilde{p}^{\#H(\omega_{n+1}\ldots\omega_N)}\tilde{q}^{\#T((\omega_{n+1}\ldots\omega_N)}[c_1X(\omega_1\ldots\omega_N) + c_2Y(\omega_1\ldots\omega_N]\\
&= c_1 \sum_{\omega_{n+1}\ldots\omega_N} \tilde{p}^{\#H(\omega_{n+1}\ldots\omega_N)}\tilde{q}^{\#T(\omega_{n+1}\ldots\omega_N)}X(\omega_1\ldots\omega_N)\\
&+ c_2 \sum_{\omega_{n+1}\ldots\omega_N} \tilde{p}^{\#H(\omega_{n+1}\ldots\omega_N)}\tilde{q}^{\#T(\omega_{n+1}\ldots\omega_N)}Y(\omega_1\ldots\omega_N)\\
&= c_1\widetilde{\mathbb{E}}_n[X](\omega_1\ldots\omega_n) + c_2\widetilde{\mathbb{E}}_n[Y](\omega_1\ldots\omega_n)
\end{aligned}$$

Thus $\widetilde{\mathbb{E}}_n[c_1X + c_2Y] = c_1\widetilde{\mathbb{E}}_n[X] + c_2\widetilde{\mathbb{E}}_n[Y]$.

Property 2. Note the following:

$$\begin{aligned}
\widetilde{\mathbb{E}}_n[XY](\omega_1\ldots\omega_n) &= \sum_{\omega_{n+1}\ldots\omega_N} \tilde{p}^{\#H(\omega_{n+1}\ldots\omega_N)}\tilde{q}^{\#T((\omega_{n+1}\ldots\omega_N)}X(\omega_1\ldots\omega_n)Y(\omega_1\ldots\omega_n\omega_{n+1}\ldots\omega_N)\\
&= X(\omega_1\ldots\omega_n)\sum_{\omega_{n+1}\ldots\omega_N} \tilde{p}^{\#H(\omega_{n+1}\ldots\omega_N)}\tilde{q}^{\#T((\omega_{n+1}\ldots\omega_N)}Y(\omega_1\ldots\omega_n\omega_{n+1}\ldots\omega_N)\\
&= X(\omega_1\ldots\omega_n)\widetilde{\mathbb{E}}_n[Y](\omega_1\ldots\omega_n\omega_{n+1}\ldots\omega_N).
\end{aligned}$$

Thus, $\widetilde{\mathbb{E}}_n[XY] = X \cdot \widetilde{\mathbb{E}}_n[Y]$

Property 3. Let $Z = \widetilde{\mathbb{E}}_m[X]$. It follows that Z depends only on $\omega_1\omega_2\ldots\omega_m$. Observe the following:

$$\begin{aligned}
\widetilde{\mathbb{E}}_n\left[\widetilde{\mathbb{E}}_m[X]\right](\omega_1\ldots\omega_n) &= \widetilde{\mathbb{E}}_n[Z](\omega_1\ldots\omega_n) \\
&= \sum_{\omega_{n+1}\ldots\omega_N} \tilde{p}^{\#H(\omega_m\ldots\omega_N)}\tilde{q}^{\#T((\omega_m\ldots\omega_N)}Z(\omega_1\ldots\omega_n\omega_m\ldots\omega_m) \\
&= \sum_{\omega_{n+1}\ldots\omega_m} \tilde{p}^{\#H(\omega_m\ldots\omega_N)}\tilde{q}^{\#T((\omega_m\ldots\omega_N)}Z(\omega_1\ldots\omega_m) \\
&\times \sum_{\omega_{m+1}\ldots\omega_N} \tilde{p}^{\#H(\omega_m\ldots\omega_N)}\tilde{q}^{\#T(\omega_m\ldots\omega_N)} \\
&= \sum_{\omega_{n+1}\ldots\omega_m} \tilde{p}^{\#H(\omega_{n+1}\ldots\omega_m)}\tilde{q}^{\#T((\omega_{n+1}\ldots\omega_m)}Z(\omega_1\ldots\omega_m) \\
&= \sum_{\omega_{n+1}\ldots\omega_m} \tilde{p}^{\#H(\omega_{n+1}\ldots\omega_m)}\tilde{q}^{\#T((\omega_{n+1}\ldots\omega_m)} \\
&\quad \sum_{\omega_{m+1}\ldots\omega_N} \tilde{p}^{\#H(\omega_{m+1}\ldots\omega_N)}\tilde{q}^{\#T((\omega_{m+1}\ldots\omega_N)}X(\omega_1\ldots\omega_N) \\
&= \sum_{\omega_{n+1}\ldots\omega_N} \tilde{p}^{\#H(\omega_{n+1}\ldots\omega_N)}\tilde{q}^{\#T((\omega_{n+1}\ldots\omega_N)}X_(\omega_1\ldots\omega_N) \\
&= \widetilde{\mathbb{E}}_n[X](\omega_1\ldots\omega_N)
\end{aligned}$$

Consequently, $\widetilde{\mathbb{E}}_n\left[\widetilde{\mathbb{E}}_m[X]\right] = \widetilde{\mathbb{E}}_n[X]$.

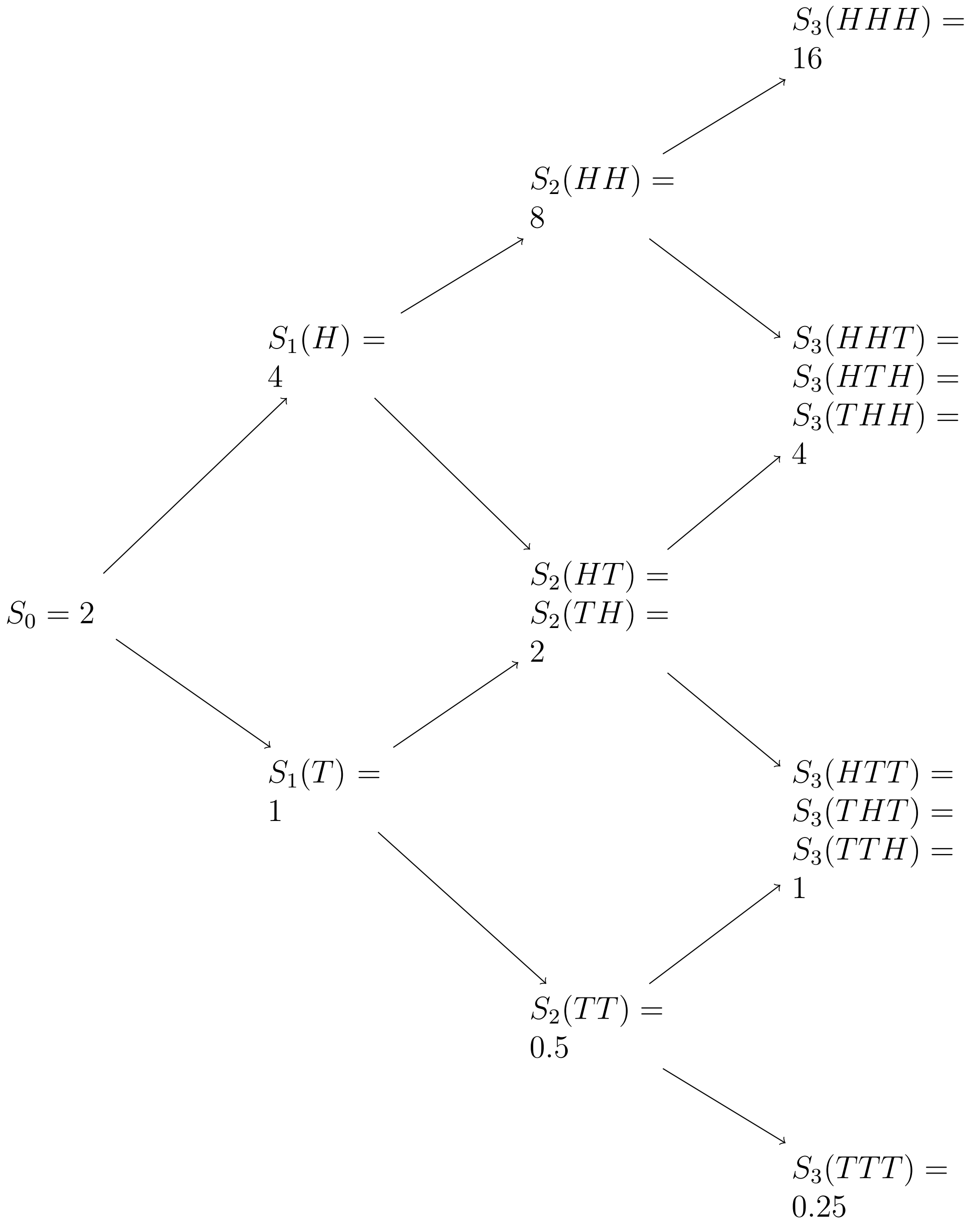
Property 4. Note that

$$\begin{aligned}
\widetilde{\mathbb{E}}_n[X] &= \sum_{\omega_{n+1}\ldots\omega_N} \tilde{p}^{\#H(\omega_{n+1}\ldots\omega_N)}\tilde{q}^{\#T(\omega_{n+1}\ldots\omega_N)}X(\omega_{n+1}\ldots\omega_N) \\
&= \sum_{\omega_1\ldots\omega_n} \tilde{p}^{\#H(\omega_1\ldots\omega_n)}\tilde{q}^{\#T(\omega_1\ldots\omega_n)} \\
&\times \sum_{\omega_{n+1}\ldots\omega_N} \tilde{p}^{\#H(\omega_{n+1}\ldots\omega_N)}\tilde{q}^{\#T(\omega_{n+1}\ldots\omega_N)}X(\omega_{n+1}\ldots\omega_N) \\
&= \sum_{\omega_1\ldots\omega_n} \tilde{p}^{\#H(\omega_1\ldots\omega_n)}\tilde{q}^{\#T(\omega_1\ldots\omega_n)}X(\omega_1\ldots\omega_n) \\
&= \widetilde{\mathbb{E}}X
\end{aligned}$$

Therefore, $\widetilde{\mathbb{E}}_n[X] = \widetilde{\mathbb{E}}X$ ■

We will conclude this section by looking at a three-period model to exemplify the properties given by Theorem 7.

Example 4. Assume $\tilde{p} = \frac{2}{3}$ and $\tilde{q} = \frac{1}{3}$ and let the model be given by the following binomial tree:



1. Linearity:

$$\begin{aligned}
\widetilde{\mathbb{E}}_1[S_2](H) + \widetilde{\mathbb{E}}_1[S_3](H) &= \big(\frac{2}{3}\cdot 8 + \frac{1}{3}\cdot 2\big) + \big(\frac{4}{9}\cdot 16 + \frac{2}{9}\cdot 4 + \frac{2}{9}\cdot 4 + \frac{1}{9}\cdot 1\big) \\
&= 6 + 9 \\
&= 15 \\
&= \frac{4}{9}(8+16) + \frac{2}{9}(8+4) + \frac{2}{9}(2+4) + \frac{1}{9}(2+1) \\
&= \widetilde{\mathbb{E}}_1[S_2 + S_3](H).
\end{aligned}$$

2. Taking out what is known:

$$\begin{aligned}
\widetilde{\mathbb{E}}_1[S_1 S_2](H) &= \frac{2}{3}\cdot 32 + \frac{1}{3}\cdot 8 \\
&= 24 \\
&= 4 \cdot 6 \\
&= S_1(H)\widetilde{\mathbb{E}}_1[S_2](H).
\end{aligned}$$

3. Iterated conditioning:

$$\begin{aligned}
\widetilde{\mathbb{E}}_1\left[\widetilde{\mathbb{E}}_2[S_3]\right](H) &= \frac{2}{3}\widetilde{\mathbb{E}}_2[S_3](HH)) + \frac{1}{3}\widetilde{\mathbb{E}}_2[S_3](HT) \\
&= \frac{2}{3}\cdot 12 + \frac{1}{3}\cdot 3 \\
&= 9 \\
&= \frac{4}{9}\cdot 16 + \frac{2}{9}\cdot 4 + \frac{2}{9}\cdot 4 + \frac{1}{9}\cdot 1 \\
&= \widetilde{\mathbb{E}}_1[S_3](H).
\end{aligned}$$

4. Independence:

$$\begin{aligned}
\widetilde{\mathbb{E}}_1\left[\frac{S_2}{S_1}\right](H) &= \frac{2}{3}\cdot\frac{S_2(HH)}{S_1(H)} + \frac{1}{3}\cdots\frac{S_2(HT)}{S_1(H)} \\
&= \frac{2}{3}\cdot 2 + \frac{1}{3}\cdot\frac{1}{2} \\
&= \frac{3}{2}
\end{aligned}$$

However $\widetilde{\mathbb{E}}_1\left[\frac{S_2}{S_1}\right]$ is equivalent to u if the second toss is H or d if the second toss is T. Thus, $\widetilde{\mathbb{E}}_1\left[\frac{S_2}{S_1}\right]$ is independent of the first coin toss, and we have that

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[\frac{S_2}{S_1}\right] &= \frac{2}{3}\cdot 2 + \frac{1}{3}\cdot\frac{1}{2} \\
&= \frac{3}{2}
\end{aligned}$$

for any ω_1.

3.2 Martingales

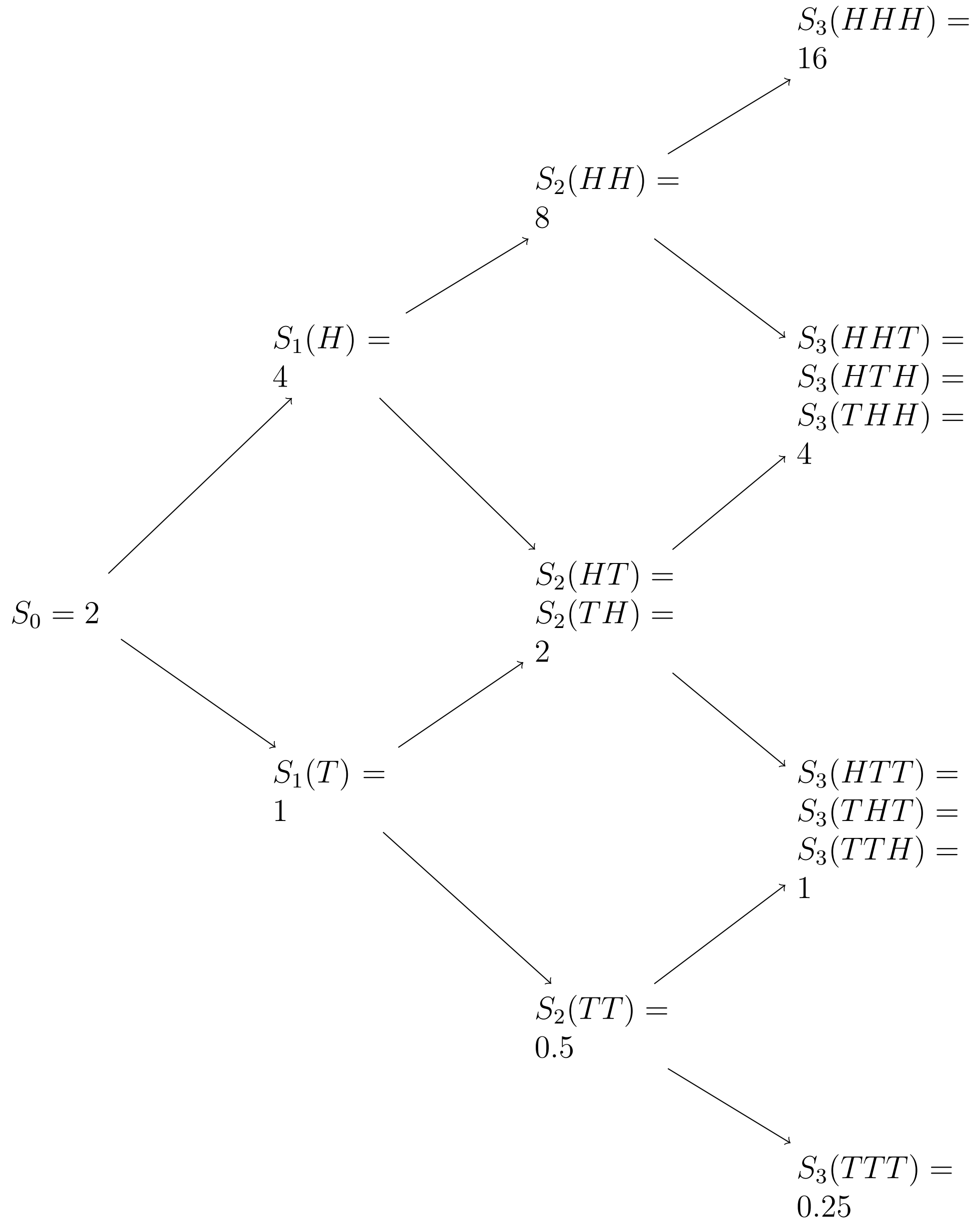
As we will observe in this section, the discounted stock price, the discounted wealth process, and the discounted price of the derivative security are all martingales under risk neutral probabilities. In general, if a random variable, M is a martingale, then the best prediction for the future value of M is its present value. We give a more rigorous definition of Martingales below:

Definition 3. Consider the binomial asset-pricing model. Let $M_0, M_1, \ldots, M_N$ be a sequence of random variables, with each M_n depending only on the first n coin tosses (and M_0 constant). Such a sequence of random variables is called a stochastic process. If

$$M_n = \widetilde{\mathbb{E}}_n[M_{n+1}], n = 0, 1, \ldots, N-1$$

we say this process is a martingale.

Example 5. Let $\tilde{p} = \frac{1}{3}$ and $\tilde{q} = \frac{2}{3}$ and assume the following binomial tree model:



Note that

$$\begin{aligned}
\widetilde{\mathbb{E}}_0[S_1] &= \tilde{p}uS_0 + \tilde{q}dS_0 \\
&= \frac{1}{3} \cdot 2 \cdot 2 + \frac{2}{3} \cdot \frac{1}{2} \cdot 2 \\
&= \frac{4}{3} + \frac{2}{3} \\
&= 2 \\
&= S_0
\end{aligned}$$

Thus $S_0 = \widetilde{\mathbb{E}}_0[S_1]$. Similarly,

$$\begin{aligned}
\widetilde{\mathbb{E}}_2[S_2](H) &= \tilde{p}uS_1(H) + \tilde{q}dS_1(H) \\
&= \frac{1}{3} \cdot 2 \cdot 4 + \frac{2}{3} \cdot \frac{1}{2} \cdot 4 \\
&= \frac{8}{3} + \frac{4}{3} \\
&= 4 \\
&= S_1(H)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\mathbb{E}}_1[S_2](T) &= \tilde{p}uS_1(T) + \tilde{q}dS_1(T) \\
&= \frac{1}{3} \cdot 2 \cdot 1 + \frac{2}{3} \cdot \frac{1}{2} \cdot 1 \\
&= \frac{2}{3} + \frac{1}{3} \\
&= 1 \\
&= S_1(T)
\end{aligned}$$

So we have that $S_1 = \widetilde{\mathbb{E}}_1[S_2]$ regardless of the outcome of the toss at time one. We may continue this for each node of our binomial tree, and find that $S_n = \widetilde{\mathbb{E}}_n[S_{n+1}]$ for all n such that $0 \leq n \leq 3$.

Theorem 3 *Consider the general binomial model with $0 < d < 1 + r < u$. Let the risk-neutral probabilities be given by*

$$\tilde{p} = \tfrac{1+r-d}{u-d} \text{ and } \tilde{q} = \tfrac{u-1-r}{u-d}$$

Then under risk-neutral measure, the discounted stock price is a martingale. That is,

$$\tfrac{S_n}{(1+r)^n} = \widetilde{\mathbb{E}}_n[\tfrac{S_{n+1}}{(1+r)^{n+1}}]$$

holds for every sequence of coin tosses.

Proof. Observe that $\frac{S_{n+1}}{S_n}$ depends only on the first $(n+1)$ coin tosses

$$\begin{aligned}
\widetilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1+r)^{n+1}}\right] &= \widetilde{\mathbb{E}}_n\left[\frac{S_n}{(1+r)^{n+1}} \cdot \frac{S_{n+1}}{S_n}\right] \\
&= \frac{S_n}{(1+r)^{n+1}}\widetilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{S_n}\right] \quad \textit{Since } S_n \textit{ is known at } n \textit{ and } 1+r \textit{ is independent from } n \\
&= \frac{S_n}{(1+r)^n} \cdot \frac{1}{(1+r)}\widetilde{\mathbb{E}}_n\frac{S_{n+1}}{S_n} \\
&= \frac{S_n}{(1+r)^n} \cdot \frac{1}{(1+r)} \cdot \frac{\widetilde{\mathbb{E}}_n S_{n+1}}{S_n} \\
&= \frac{S_n}{(1+r)^n} \cdot \frac{1}{(1+r)} \cdot \frac{\tilde{p}uS_n + \tilde{q}dS_n}{S_n} \\
&= \frac{S_n}{(1+r)^n} \cdot \frac{\tilde{p}u + \tilde{q}d}{1+r} \\
&= \frac{S_n}{(1+r)^n} \quad \textit{Since } \tilde{p}u + \tilde{q}d = 1+r
\end{aligned}$$

Therefore, we have that $\frac{S_n}{(1+r)^n} = \widetilde{\mathbb{E}}_n[\frac{S_{n+1}}{(1+r)^{n+1}}]$. Consequently, the discounted stock price $\frac{S_n}{(1+r)^n}$ is a martingale. ■

Theorem 4 *Consider the binomial model with N periods. Let $\Delta_0, \Delta_1, \ldots \Delta_{N-1}$ be an adapted portfolio process, let $X_0 \in \mathbb{R}$ and let the wealth process $X_1, \ldots X_N$ be generated recursively by $X_n = \Delta_{n-1}S_n + (1+r)(X_{n-1} - \Delta_{n-1}S_{n-1})$ where $n = 0, 1, \ldots N-1$. Then the discounted wealth proceed $\frac{X_n}{(1+r)^n}$, $n, 1 \ldots, N$, is a martingale under the risk-neutral measure. That is,*

$$\frac{X_n}{(1+r)^n} = \widetilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right], n = 0, 1, \ldots, N-1$$

Proof. Observe the following:

$$\begin{aligned}
\widetilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] &= \widetilde{\mathbb{E}}_n\left[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}} + \frac{X_n - \Delta_n S_n}{(1+r)^n}\right] \\
&= \widetilde{\mathbb{E}}_n\left[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}}\right] + \widetilde{\mathbb{E}}_n\left[\frac{X_n - \Delta_n S_n}{(1+r)^n}\right] \quad \textit{By Linearity} \\
&= \Delta_n \widetilde{\mathbb{E}}_n\left[\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}}\right] + \frac{X_n - \Delta_n S_N}{(1+r)^n} \quad \textit{Taking out what is known} \\
&= \Delta_n \frac{S_n}{(1+r)^n} + \frac{X_n - \Delta_n S_N}{(1+r)^n} \quad \textit{By Theorem 3} \\
&= \frac{X_n}{(1+r)^n}
\end{aligned}$$

Thus, $\frac{X_n}{(1+r)^n} = \widetilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right], n = 0, 1, \ldots, N-1$. Consequently, the discounted wealth process is a martingale. ■

Theorem 5 *Consider an N period binomial asset-pricing model with $0 < d < 1+r < u$ and with risk neutral probability measure $\widetilde{\mathbb{P}}$. Let V_N be a random variable depending on the coin tosses. For n between 0 and N the price of the derivative security at time n is given by the risk-neutral pricing formula $V_n = \widetilde{\mathbb{E}}_n\left[\frac{V_n}{(1+r)^{N-n}}\right]$. Furthermore, the discounted price of the derivative security is a martingale under $\widetilde{\mathbb{P}}$. That is,*

$$\frac{V_n}{(1+r)^n} = \widetilde{\mathbb{E}}_n\left[\frac{V_{n+1}}{(1+r)^{n+1}}\right], n = 0, 1, \ldots N-1.$$

Proof. Recall than $X_N(\omega_1, \ldots, \omega_N) = V_N(\omega_1, \ldots, \omega_N)$ for all sequences $(\omega_1, \ldots \omega_N)$ by Theorem 1. Thus, it follows that $\frac{X_n}{(1+r)^n} = \frac{V_n}{(1+r)^n}$ for $n \in \mathbb{Z}^+$ such that $0 \leq n \leq N$. Additionally, The discounted wealth equation is a martingale, so we have that $\frac{X_n}{(1+r)^n} = \widetilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right], n = 0, 1, \ldots, N-1$. Observe that

$$\begin{aligned}
\frac{V_n}{(1+r)^n} &= \frac{X_n}{(1+r)^n} \\
&= \widetilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] \\
&= \widetilde{\mathbb{E}}_n\left[\frac{V_{n+1}}{(1+r)^{n+1}}\right].
\end{aligned}$$

Consequently, $\frac{V_n}{(1+r)^n} = \widetilde{\mathbb{E}}_n\left[\frac{V_{n+1}}{(1+r)^{n+1}}\right]$ and the discounted price of the derivative security is a martingale. ■

Under risk-neutral measures, we have defined our stock price, wealth, and derivative payoff in such a way that, when discounted for money market returns, are martingales. This aspect allows an agent to use the current value of a random variable as the best prediction for the variables future value.

4 Interest-Rate Dependent Assets

A fixed income asset is an asset thats value is dependent on fluctuating interest rates. One common fixed income asset is a zero coupon bond, which pays off a set amount, known as its face value, at a specified time of maturity. A yield is the constant interest rate that would be required from the initial period until the period of maturity in order for the bond to reach its full face value.

4.1 Interest Rate Processes

Definition 4. Let Ω be the set of 2^N possible outcomes $\omega_1\omega_2\dots\omega_N$ under the risk neutral probability measure $\widetilde{\mathbb{P}}$. The **interest rate process** is a sequence of random variables

$$R_0, R_1, \dots R_{N-1}$$

where R_n depends on the first n coin tosses and R_0 is not random.

Definition 5. The **discount rate process** is given by

$$D_n = \frac{1}{(1+R_0)(1+R_1)\cdots(1+R_{n-1})}, n = 1, 2, \dots N$$

and $D_0 = 1$.

4.2 Zero-Coupon Bonds

The price at $t = 0$ for a zero-coupon bond pays 1 at a maturity time m is given by

$$B_{0,m} = \tilde{\mathbb{E}}[D_m].$$

We can solve for the yield of the bond, y_m to find that the rate needed to reach full maturity at time m is given by

$$y_m = \left(\frac{1}{B_{0,m}}\right)^{\frac{1}{m}} - 1.$$

Definition 6. Let $\widetilde{\mathbb{P}}$ be to probability measure on Ω, which is the space of all possible sequences given by N coin tosses where each sequence has strictly positive probability. Assume $1 \leq n \leq N - 1$ and let $\bar{\omega_1}, \dots, \bar{\omega_N}$ be a sequence of N tosses. Define

$$\widetilde{\mathbb{P}}\{\omega_{n+1} = \bar{\omega}_{n+1}, \dots \omega_N = \bar{\omega_N} | \omega_1 = \bar{\omega}_1, \dots \omega_n = \bar{\omega}_n\} = \frac{\widetilde{\mathbb{P}}\{\bar{\omega}_1, \dots \bar{\omega}_n, \bar{\omega}_{n+1}, \dots \bar{\omega}_n\}}{\widetilde{\mathbb{P}}\{\omega_1 = \bar{\omega}_1, \dots, \omega_n = \bar{\omega}_n\}}.$$

Let X be a random variable. The **conditional expectation of X based on the information at time n** is given by

$$\widetilde{\mathbb{E}}_n[X](\bar{\omega}_1, \dots, \bar{\omega}_n) = \sum_{\bar{\omega}_{n+1}, \dots, \bar{\omega}_N} X(\bar{\omega}_1, \dots, \bar{\omega}_n \bar{\omega}_{n+1} \dots \bar{\omega}_N) \times \widetilde{\mathbb{P}}\{\omega_{n+1} = \bar{\omega}_{n+1}, \dots \omega_N = \bar{\omega}_N | \omega_1 = \bar{\omega}_1, \dots \omega_n = \bar{\omega}_n\}.$$

Furthermore, we have $\widetilde{\mathbb{E}}_0[X] = \widetilde{\mathbb{E}}X$ and $\widetilde{\mathbb{E}}_N[X] = X$.

Observe that Definition 6 is a modification of Definition 2 which allows for coin tosses that are not independent. Additionally, the four properties of conditional expectations proved in Theorem 2 of section 3.1 hold under this definition as well.

Definition 7. Let $\widetilde{\mathbb{P}}$ be to probability measure on Ω, which is the space of all possible sequences given by N coin tosses where each sequence has strictly positive probability. For $0 \leq n \leq m \leq N$ the **price at time n of the zero-coupon bond maturing at time** m is defined to be

$$B_{n,m} = \tilde{\mathbb{E}}_n[\tfrac{D_m}{D_n}].$$

It is important to note that this definition is chosen in order to ensure the discounted zero-coupon bond prices are martingales, which we will formalize in the following theorem.

Theorem 6 *Let $\widetilde{\mathbb{P}}$ be to probability measure on Ω, which is the space of all possible sequences given by N coin tosses where each sequence has strictly positive probability. For $0 \leq n \leq m \leq N$, the Zero-Coupon Bond Price given by*

$$B_{n,m} = \tilde{\mathbb{E}}_n[\tfrac{D_m}{D_n}]$$

is a martingale.

Proof. Let $0 \leq n \leq m \leq N$ and zero-coupon bonds be priced such that $B_{n,m} = \tilde{\mathbb{E}}_n[\frac{D_m}{D_n}]$. Since D_n is known at time n, we then have that

$$D_n B_{n,m} = \tilde{\mathbb{E}}_n[D_m].$$

By applying iterated conditioning where $0 \leq k \leq n \leq m$,

$$\tilde{\mathbb{E}}_k[D_n B_{n,m}] = \tilde{\mathbb{E}}_k[\tilde{\mathbb{E}}_n[D_m]] = \tilde{\mathbb{E}}_k[D_m] = D_k B_{k,m}.$$

Thus, $\tilde{\mathbb{E}}_k[D_n B_{n,m}] = D_k B_{k,m}$ and we conclude that the zero-coupon bond price is a martingale. ■

In order to quantify the wealth of an agent investing in zero-coupon bonds, we must consider the agents zero-coupon bonds paying off in the current period, the zero-coupon bonds that the agent holds which will payoff at a future time, and the amount that the agent has invested in the money market. Considering a portfolio of zero-coupon bonds of all maturities, the wealth of the portfolio at time $n+1$, given and initial wealth of X_n is given by

$$X_{n+1} = \Delta_{n,n+1} + \sum_{m=n+2}^{N} \Delta_{n,m} B_{n+1,m} + (1+R_n)(X_n - \sum_{m=n+1}^{N} \Delta_{n,m} B_{n,m}).$$

In the wealth equation above, $\Delta_{n,n+1}$ is the payoff of zero-coupon bonds which matured in the period $n+1$. The second term, $\sum_{m=n+2}^{N} \Delta_{n,m} B_{n+1,m}$ is the collective value of bonds which will mature in future periods. Finally, $(1+R_n)(X_n - \sum_{m=n+1}^{N} \Delta_{n,m} B_{n,m})$ gives the money market returns multiplied by the value of remaining wealth that was not invested in bonds and instead earns money market returns.

Theorem 7 *Regardless of how $\Delta_{n,m}$ is chosen, the discount wealth process $D_n X_n$ is a martingale under $\widetilde{\mathbb{P}}$.*

Proof. Note that $\Delta_{n,m}$ may depend only on the first n tosses and D_{n+1} depends on $R_0, R_1, \dots R_n$ for the first n tosses. Additionally, we have that $B_{n,m}$ is a martingale. It follows that

$$\begin{aligned}
\widetilde{\mathbb{E}}[X_{n+1}] &= \Delta_{n,n+1} + \sum_{m=n+2}^{N} \Delta_{n,m}\widetilde{\mathbb{E}}_n[B_{n+1,m}] \\
&+ (1+R_n)\left(X_n - \sum_{m=n+1}^{N} \Delta_{n,m}B_{n,m}\right), \\
&= \Delta_{n,n+1} + \sum_{m=n+2}^{N} \frac{\Delta_{n,m}}{D_{n+1}}\widetilde{\mathbb{E}}_n[D_{n+1}B_{n+1,m}] \\
&+ \frac{D_n}{D_{n+1}}\left(X_n - \sum_{m=n+1}^{N} \Delta_{n,m}B_{n,m}\right), \\
&= \Delta_{n,n+1} + \sum_{m=n+2}^{N} \frac{\Delta_{n,m}}{D_{n+1}}D_nB_{n,m} + \frac{D_n}{D_{n+1}}X_n \\
&- \frac{D_n}{D_{n+1}}\sum_{m=n+1}^{N} \Delta_{n,m}B_{n,m}, \\
&= \Delta_{n,n+1} + \frac{D_n}{D_{n+1}}X_n - \frac{D_n}{D_{n+1}}\Delta_{n,n+1}B_{n,n+1}.
\end{aligned}$$

However, $B_{n,n+1}$ depends only on the first n tosses so it follows that $B_{n,n+1} = \widetilde{\mathbb{E}}_n[\frac{D_{n+1}}{D_n}] = \frac{D_{n+1}}{D_n}$. Thus, we have that

$$\widetilde{\mathbb{E}}[X_{n+1}] = \frac{D_n}{D_{n+1}}X_n.$$

Since D_{n+1} is know at time n, we pull out what is known and conclude that

$$\widetilde{\mathbb{E}}[D_{n+1}X_{n+1}] = D_nX_n.$$

Hence, the discounted wealth process is a martingale. ■

According to the risk-neutral pricing formula, if $0 \leq n \leq m \leq N$, the price of a derivative security paying V_m at time m is given by

$$V_n = \frac{1}{D_n}\widetilde{\mathbb{E}}_n[D_mV_m].$$

Depending on the payoff definition for each particular bond related derivative security, the above formula can be used to recursively price a security in a similar fashion to the examples in Section 2.3.

5 References

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3. Multi-period model:
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