Using Mathematics to Optimize Evacuation Routes

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Abstract

Most evacuation routes are determined by the shortest path from the occupied room to an exit. This design fails to take into account congestion caused by the number of evacuees. We model Georgia College's Arts and Sciences building with a modified minimum cost flow network and use non-linear and linear programming techiniques to determine optimal evacuation routes with and without congestion respectively.

1 Introduction

Most escape routes are determined by the shortest distance to an exit. These paths do not take into account the size of the hallways or the amount of people using the routes. There may be situations in which it is more efficient and therefore safer to travel a further distance encumbered by fewer people when trying to evacuate. This paper considers congestion in determining the optimal evacuation routes, specifically in Georgia College's Arts and Sciences building (A&S).

A flow network is a directed graph with specific types of nodes: transshipments, sources, and sinks. Goods are transferred through these nodes, moving from sources to sinks via weighted edges. The weights on the edges typically represent the cost of traveling along that path. The goods originate at the sources, meaning the sources are the supply nodes whereas the sinks are the final destination and only have a demand. The transshipment nodes are between the sources and sinks and have no supply or demand. The solution to a network flow problem is the set of optimal routes between nodes. This optimization can take multiple forms such as finding the maximum flow, minimum cost, or shortest path [4]. This specific problem is modeled as a minumum cost network flow problem.

The solution to minimum cost network flow problems is the set of routes which accomplish the transference of all goods from the supply nodes to the sink nodes with the smallest cost [2]. Mathematically, this is accomplished using the following functions:

$$\text{Minimize:} \quad \sum_{(i,j)\in H} C_{ij}X_{ij} \tag{1}$$

$$\text{Subject to:} \quad \sum_{\{j:(i,j)\in H\}} X_{ij} - \sum_{\{j:(j,i)\in H\}} X_{ji} = b(i) \quad \text{for all nodes} \quad i, \tag{2}$$

$$l_{ij} \leq X_{ij} \leq u_{ij}. \tag{3}$$

The objective function (1) being minimized is a sum across all edges (i, j) in the set of edges H. It is a sum of the product of the cost, denoted C_{ij}, and the amount of goods on route (i, j), written X_{ij}. Note the cost is known, and the solution will be in terms of X_{ij}. The constraint (2) is the flow constraint on each node i. In its most basic form, this constraint states that the difference between the flow out of node i and the flow into node i is equal to the supply or demand of the node. Rearranged, this constraint can be stated

$$\text{supply} + \text{flow in} = \text{demand} + \text{flow out}.$$

If $b_i < 0$, then i is a demand node. If $b_i > 0$, then i is a supply node. Lastly, if $b_i = 0$ then i is a transshipment node. Also, there is a load constraint (3) on each edge (i, j) which states that the amount of goods on (i, j) cannot be below the minimum capacity of the edge, l_{ij}, or above the maximum capacity, u_{ij} [3].

2 Model

To model an evacuation of the Arts and Sciences building, each room as well as the doors to exits, and the exits themselves are represented as nodes in a network. The edges of the graph denote the paths between the rooms and exits. The weight on an edge is the time in seconds required for one person to travel the length of that hallway in A&S. This model implements time as the cost of movement along each edge. Hence the goal of this model is to the find evacuation routes which require the least amount of time to get all of the occupants out of the building. Mathematically, we minimize the function

$$\sum_{(i,j)\in H} C_{ij}X_{ij}$$

where C_{ij} is the time from node i to node j, and X_{ij} is the number of people moving from node i to node j. The n nodes are divided into three groups: sinks, sources, and transshipment nodes. Due to the application of this specific minimum cost flow problem, we let E be the set of exit nodes. The sources will be any room of A&S that typically has occupants. These are classrooms, offices, conference rooms, etc., denoted N_i. Due to the likelihood that occupants will need to bypass some exits in order to use others, transshipment nodes are necessary at each exit. Hence the doors to an exit are modeled as the transshipment nodes, represented by D_i. Note that

$$E + N + D = n.$$

This specific model utilizes the standard constraints on the source and transshipment nodes. For all source nodes N_i,

$$\sum_{\{j:(j,N_i)\in H\}} X_{jN_i} + \text{supply of } N_i = \sum_{\{j:(N_i,j)\in H\}} X_{N_ij}. \tag{4}$$

Similarly, since transshipment nodes have no demand or supply, the constraint on transshipment nodes D_i is

$$\sum_{\{j:(j,D_i)\in H\}} X_{jD_i} = \sum_{\{j:(D_i,j)\in H\}} X_{D_ij}. \tag{5}$$

The supply is the number of people occupying that node when the evacuation is initiated. There is no demand in the rooms of A&S since people are not required to enter a room as they pass. Thus the flow into a node is the number of people that pass that area of the building during the evacuation process.

Unlike most minimum cost network flow models, the specific demand of each sink node is unknown. However since every person must be evacuated from the building, the sum of the demands at the exits is equal to the number of occupants of the Arts and Sciences building. The adjusted constraint on the exits is defined by

$$\sum_{i\in E}\left(\sum_{\{j:(E_i,j)\in H\}} X_{E_ij} - \sum_{\{j:(j,E_i)\in H\}} X_{jE_i}\right) = \sum_{i\in E} b(i). \tag{6}$$

These constraints are implemented in each model used.

2.1 Assumptions

We model the worst case scenario. Hence each room is assumed to be filled to maximum capacity. These capacities were found by checking each room and recording the number of student and teacher desks. In the majority of rooms, this information is posted. Offices are assumed to have one occupant; department chair offices have two. Other rooms, such as supply closets, are assumed to be unoccupied. At the start of the evacuation process everyone is in the rooms. This model works on the assumption that there is no panic during the evacuation, that is everyone is moving at a steady speed with no running or trampling, and everyone is leaving the building. The time for a student to walk the length of each hallway at a brisk pace when the path is unoccupied was found empirically, resulting in the assumed walking rate for each person of 4.4 feet per second. Each flight of stairs is 13 steps. It is assumed that it takes 0.5 seconds to traverse each step and 1.14 seconds to cross a five foot landing. The design of the model takes on the assumption that there is no cost in leaving the classrooms and that the occupants of each classroom can take different routes. Also, every route is available, meaning the cause of evacuation has not prevented access to any part of the building.

2.2 No Congestion Models

As an example, we solve a simple model which has one story, four classrooms, two transshipment nodes, and two exits. For this model, it is assumed there is no congestion in the hallways or at the exits. The supply of the N_i classrooms, D_i transshipment nodes, and E_i exits are labeled as well as the cost between nodes. The network is shown in figure 1.

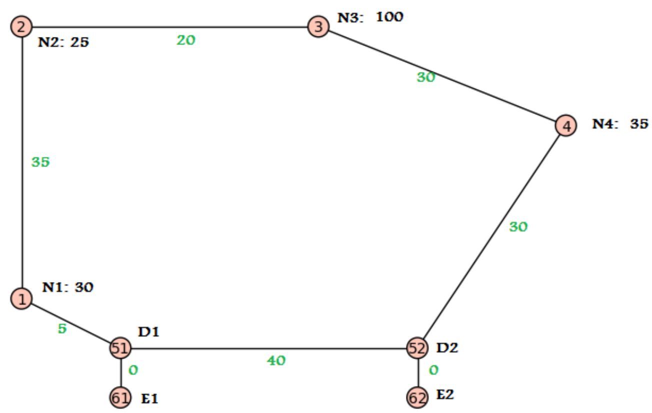


Figure 1: Small Model

This model utilizes the linear objective function and is thus solved with linear programming. After solving this simple model through the program AMPL, we checked the solution by hand. The full linear programming model is seen below [1].

$$\begin{aligned}\text{Minimize:} \sum_{(i,j)\in H} C_{ij}X_{ij} =& 35X_{N_1N_2} + 5X_{D_1N_1} + 20X_{N_2N_3} + 30X_{N_3N_4} + 30X_{N_4D_2} + 40X_{D_2D_1} + 5X_{N_1D_1} \\ &+ 40X_{D_1D_2} + 30X_{D_2N_4} + 30X_{N_4N_3} + 20X_{N_3N_2} + 35X_{N_2N_1}\end{aligned}$$

$$\begin{aligned}\text{subject to: } & X_{N_1N_2} + X_{N_1D_1} - (X_{N_2N_1} + X_{D_1N_1}) = 30, \\ & X_{N_2N_1} + X_{N_2N_3} - (X_{N_1N_2} - X_{N_3N_2}) = 25, \\ & X_{N_3N_2} + X_{N_3N_4} - (X_{N_2N_3} - X_{N_4N_3}) = 100, \\ & X_{N_4D_2} + X_{N_4N_3} - (X_{N_2N_4} - X_{N_3N_4}) = 35, \\ & X_{D_2N_4} + X_{D_2E_2} + X_{D_2D_1} = X_{N_4D_2} + X_{E_2D_2} + X_{D_1D_2}, \\ & X_{D_1N_1} + X_{D_1E_1} + X_{D_1D_2} = X_{N_1D_1} + X_{E_1D_1} + X_{D_2D_1}, \\ & X_{E_1D_1} - X_{D_1E_1} + X_{E_2D_2} - X_{D_2E_2} = 190, \\ & 0 \le X_{ij} \le \infty, \text{ for all } (i,j) \in H.\end{aligned}$$

The second model addressed involves all three floors of Arts and Sciences. This model works with the same assumptions as before and still includes no congestion and is hence linear. The network is seen below.

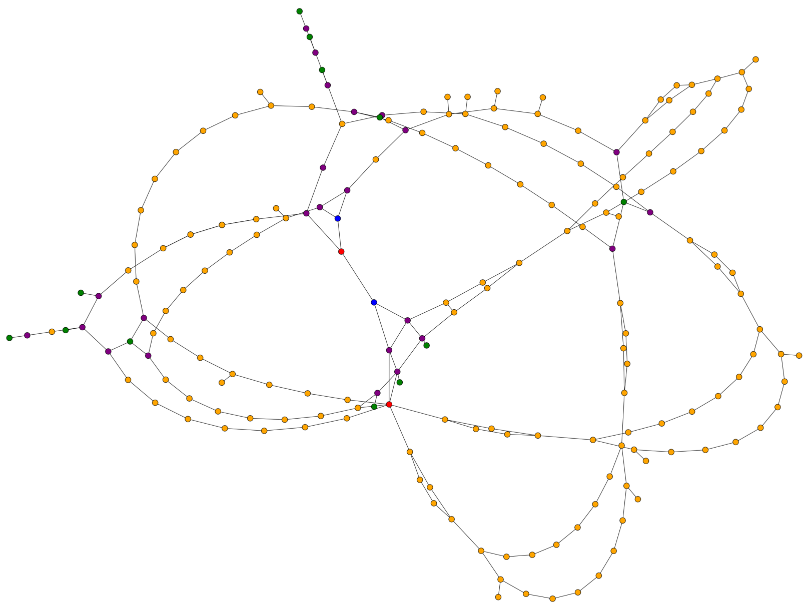


Figure 2: Arts and Science Building

Modeled in this manner, there are 194 nodes representing A&S. The blue nodes are the courtyard doors; the red are the two staircases in the building that do not lead directly to exits. Purple nodes are transshipment nodes. Green nodes are exits, and the yellow nodes are the source nodes. This network has 438 variables, representing the various routes that can be taken. The objective function is subject to 178 linear constraints. Since both the objective function and the constraints are linear, the solution can be found via linear programming. The general model for uncongested flow is

$$\text{Minimize:} \quad \sum_{(i,j)\in H} C_{ij}X_{ij}$$

$$\text{Subject to:} \quad \sum_{\{j:(j,N_i)\in H\}} X_{jN_i} + \text{supply of } N_i = \sum_{\{j:(N_i,j)\in H\}} X_{N_ij},$$

$$\sum_{\{j:(j,D_i)\in H\}} X_{jD_i} = \sum_{\{j:(D_i,j)\in H\}} X_{D_ij},$$

$$\sum_{i\in E}\left(\sum_{\{j:(E_i,j)\in H\}} X_{E_ij} - \sum_{\{j:(j,E_i)\in H\}} X_{jE_i}\right) = 2478,$$

$$0 \le X_{ij} \le \infty \text{ for all } (i,j)\in H.$$

2.3 Congestion Models

The final set of models uses the same network as above while imposing congestion on the routes. We develop two such models, one which is integer relaxed and one which is integer constrained. Both require a function to model the congestion. This function is based on the assumption that it takes each person an extra $\frac{1}{3}$

second to traverse a path for every person that is directly ahead of them on the same route. Thus the average time for one person to traverse a route of length L congested by n people is

$$\frac{1}{n}\sum_{k=1}^{n}\left[\left(\frac{L}{4.4}\right)+\frac{1}{3}(k-1)\right] = \frac{1}{n}\left[\frac{L}{4.4}\sum_{k=1}^{n}1+\frac{1}{3}\sum_{k=1}^{n}k-\frac{1}{3}\sum_{k=1}^{n}1\right]$$

$$= \frac{1}{n}\left[\frac{L}{4.4}n+\frac{1}{3}\left(\frac{n(n+1)}{2}\right)-\frac{1}{3}n\right] \tag{7}$$

$$= \frac{L}{4.4}+\frac{n+1}{6}-\frac{1}{3}.$$

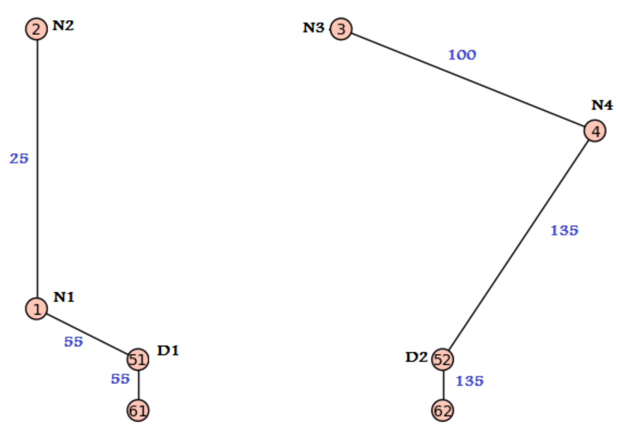
Converting equation (7) into the standard notation of a minimum cost network flow problem implies that the total time required to move n people from node i to node j is

$$\sum_{(i,j)\in H}\left(C_{ij}+\frac{X_{ij}}{6}-\frac{1}{3}\right)X_{ij}. \tag{8}$$

This is the adjusted objective function which now takes into account congestion. Note that this function is no longer linear. It is quadratic in X_{ij} which requires non-linear programming. Two models impose this congestion function. One is integer relaxed while the other is integer constrained.

3 Results

The solution to the simple model follows traditional evacuation routes. Since there is no negative consequence for having numerous people on a route, everyone travels to and uses the nearest exit. The graph below models the evacuation routes used. The labels on the edges denote the number of people using that specific route.



Note that there are 135 people using one exit while there are only 55 at the other. Using these routes, the sum total time used in evacuating the building is 8200 seconds, meaning on average it takes each person 43.16 seconds to exit.

The NEOS Server is used to find the optimal routes of the noncongested complete model [5]. After 478 iterations, the sum total time to evacuate the Arts and Sciences building with no congestion is 45708.3

seconds. This is an average of 18.42 seconds for each of the 2478 people in the building. Since this is an average, some people, such as the people closest to the exits, will be able to exit faster while others will take longer. Some very congested routes are displayed below.

Route	Number of People
$N169 \rightarrow D6$	302
$D1 \rightarrow E1$	157
$N263 \rightarrow D24$	175
$N302 \rightarrow S1$	66
$N370 \rightarrow S1$	91
$D22 \rightarrow E4$	272
$D12 \rightarrow E4$	227
$D4 \rightarrow E4$	261

In total, 157 from the third floor use a single stairwell, $S1$. This stairwell is in the northeast corner of the building and does not lead directly to an exit. Another congested stairwell is the one leading to $E4$ which is in the southwest corner, behind Terrell Hall. Altogether 760 people use at least one of the flights of stairs in that stairwell. The 157 people at exit $E1$ across from the health science building are only two feet away from another exit that only services 32 people. Similarly, all 175 people in the auditorium, $N263$, use the door within the auditorium that leads directly outside instead of entering the halls of A&S and then exiting.

The integer relaxed congestion model, solved with the Bonmin solver on the NEOS server, has an optimal solution of 166103.9 seconds [5]. This is an average of 1.12 minutes per person. Again, some will be able to exit faster and some will be slower. However since this model allows non-integer solutions, there are fractional parts of people on each route. This is entirely impractical and highly undesirable. Hence we solve a congestion model that is integer constrained, meaning there will no longer be parts of people going in multiple directions.

The integer constrained model required 9425243 iterations to find the optimal solution of a sum total of 203995.51 seconds to evacuate the building, meaning an average of 1.37 minutes per person. The number of people on each route of the integer constrained congestion model differs only slightly from the integer relaxed model.

Perhaps the most interesting routes utilized in the integer constrained congestion model involve the A&S courtyard. It is unlikely that current evacuation routes require people to use this area of the building. However, when congestion is considered, it is more efficient and therefore safer for 36 first floor occupants to cross the courtyard during their evacuation. In fact, 15 of these people ascend the courtyard stairs to exit out of the second floor. This is likely because there are five exits on the hallway that this stairway leads to on the second floor in comparison to the total of six exits on the first floor.

3.1 Analysis

This table compares the results of each of the models.

Model	Sum Total Time (s)	Average Time (s)
No Congestion	45708.3	18.42
Integer Relaxed Congestion	166103.9	67.2
Integer Constrained Congestion	203995.51	82.2

The evacuation time increases as the constraints on the model increase. Without congestion, people are able to escape the building rapidly. The average time is the same as the average cost across the routes. The congestion models result in a greater average evacuation time due to the added constraint of congestion. Since it includes the congestion constraint as well as the restriction of only producing integer results, the integer constrained model has the longest evacuation time.

In general, the congestion model includes two strategies that the noncongestion model fails to utilize. The first is that the congestion model results in people being dispersed across exits. Certain routes that demonstrate this are seen in the table below.

Route	**No Congestion**	**Congestion**
$D15 \to E7$	189	65
$D16 \to E8$	0	26
$D17 \to E9$	0	13
$D20 \to E5$	313	155
$D18 \to E10$	0	103
$D19 \to E11$	0	105
Auditorium $\to D24$	175	98
Auditorium $\to D19$	0	77

The three exits $E7, E8$, and $E9$ are side by side in a central location on the second floor of the Arts and Sciences building. When there is no congestion, the 189 people that reach this part of the building use the first door they come to in this set of exits since there is no time spent waiting and there is a cost to move to another door. However with congestion included, the people in this area of the building spread out across these three doors. Similarly, without congestion, 313 people on the second floor use $D20$ to $E5$. This involves going down 13 stairs. However, $D20$ is not far from $D18$ and $D19$ which both lead to exits without stairs. Thus when congestion is imposed, people disperse and nearly balance the number of people using these paths.

4 Conclusions

Since this model is conducted based on the worst case scenario for evacuating the Arts and Sciences building, it is applicable for any amount of occupants in the building up to maximum capacity. This model can be adapted to any building and can take into account any cause for evacuation. This project has laid the groundwork for further developments. Imposing time would be very beneficial. This would enable us to see where people are in the building at various times during the evacuation instead of only a total count of users of each route at the completion of the evacuation. Including time in this manner would also be a step toward being able to track the entire route used by a specific occupant.

The current model allows the occupants of rooms to separate and take different routes. Preventing this separation would be more realistic and easier to put into effect because people would not have to first decide who would go where out of their group; they could move together. Also, this current model does not include congestion in entering the hallway. To implement this, it would be necessary to create transshipment nodes between each source node and the hallways. A congestion function then could be developed and imposed on the movement from the source nodes and through the new transshipment nodes.

In future work, it would be beneficial to construct a model which could easily be adjusted to account for a blocked route. Meaning, the fact that the hallway between rooms 79 and 83 is on fire could be input and taken into account in the construction of routes. This model has already shown that evacuation routes should be altered to avoid unnecessary congestion and therefore be more efficient and more safe. With these recommended additions and developments, this model would be ready to present to health and safety boards. Changing evacuation routes could save lives.

Once the evacuation strategy is changed, it must be implemented. One step in this process would be to simply call attention to the change. People need to be aware that there is a new, safer method of evacuation so that they are prepared to adhere to it in an emergency. Secondly, new maps modeling the evacuation routes used in the worst case scenario should be placed in every room and hallway of every building. Developing an emergency application using new smart phone techonology would allow each person to have access to their specific route. The emergency alert system would be connected to each occupant's mobile device. The GPS would then take into account how many people are in each area and direct people accordingly. With the current advances in technology, this implementation is probable, yet first the change in evacuation routes must be made.

References

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