# Some Results on Fuzzy Matrices

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**Abstract**

A fuzzy matrix is a matrix whose entries are real numbers in the interval [0, 1]. We study properties of fuzzy matrices. Particular attention is given to the case of K-idempotent fuzzy matrices. We characterize 2-by-2 K-idempotent fuzzy matrices and n-by-n K-idempotent triangular fuzzy matrices.

*Keywords:* Fuzzy matrix, Fuzzy determinant, $K-$idempotence.

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## 1. Introduction

Fuzzy Matrix Theory was first introduced by Michael G. Thomason in 1977 as a branch of Fuzzy Set Theory, which was developed by L.A. Zadeh twelve years prior [6]. The motivation behind Zadeh's exploration of fuzzy sets was the fact that in physical reality, there exist objects that cannot be placed under clearly defined criteria of membership. For instance, Zadeh points to the "class of all real numbers which are much greater than 1" [7]. It would be impossible to precisely define such a set of real numbers, and therefore we would consider this to be a fuzzy set.

Fuzzy matrices have applications in a broad spectrum of fields. For instance, fuzzy matrices have proven very useful within the medical field. Since there is often uncertainty in information about patients, symptoms, and diagnoses, fuzzy matrices assist in more accurately representing such uncertainty while also pointing to the most likely candidate for diagnosis. Meenakshi and Kaliraja, in their work on interval valued fuzzy matrices for medical diagnosis, state that by using fuzzy matrices with sets of symptoms, diseases, and patients, we can calculate diagnosis scores both for and against respective diseases [4].

Fuzzy matrices have also been used in the agricultural field to determine crops that are the most well-suited to a specific patch of land. This takes into consideration the biophysical, economic, social, and environmental impacts of a given crop [1]. Fuzzy matrices are extremely useful in dealing with this large amount of information. Like Agriculture and Medicine, any field dealing with uncertainty in information and decision-making could possibly benefit from the use of fuzzy matrices.

This paper will be focused on fuzzy matrices and some definitions and propositions related to them. Topics will include fuzzy matrix operations, fuzzy determinants, fuzzy traces, and $K-$idempotence.

## 2. Fuzzy Matrices

**Definition 2.1.** (1) *Let $A$ be an $n \times m$ matrix defined by*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm}. \end{bmatrix}$$

*The matrix $A$ is a **fuzzy matrix** if and only if $a_{ij} \in [0,1]$ for $1 \le i \le n$ and $1 \le j \le m$. In other words, any $n \times m$ matrix $A$ is a fuzzy matrix if the elements of $A$ are in the interval [0,1]. [3]*

(2) *We define **fuzzy addition** $+$, **fuzzy multiplication** $\cdot$, and **fuzzy subtraction** $-$ as follows:*

$$\begin{aligned} a + b &= \max(a,b), \\ a \cdot b &= \min(a,b), \quad \textit{and} \\ a - b &= \begin{cases} a \text{ if } a > b \\ 0 \text{ if } a \le b. \end{cases} \end{aligned}$$

*[5]*

**Proposition 2.2.** *Let $A, B, C$ be three $n \times n$ fuzzy matrices. With the **fuzzy addition** defined in Definition 2.1, we have the following:*

(1) *$A + B = B + A$ (Commutativity),*

(2) *$(A + B) + C = A + (B + C)$ (Associativity),*

(3) *$A + 0 = 0 + A = A$ (Additive Identity).*

*Proof.* Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$, and $C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$.

(1) Observe the following:

$$\begin{aligned} A+B &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \\ &= \begin{bmatrix} \max(a_{11}, b_{11}) & \max(a_{12}, b_{12}) & \cdots & \max(a_{1n}, b_{1n}) \\ \max(a_{21}, b_{21}) & \max(a_{22}, b_{12}) & \cdots & \max(a_{2n}, b_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ \max(a_{n1}, b_{n1}) & \max(a_{n2}, b_{n2}) & \cdots & \max(a_{nn}, b_{nn}) \end{bmatrix} \end{aligned}$$

On the other hand,

$$\begin{aligned} B+A &= \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \\ &= \begin{bmatrix} \max(b_{11}, a_{11}) & \max(b_{12}, a_{12}) & \cdots & \max(b_{1n}, a_{1n}) \\ \max(b_{21}, a_{21}) & \max(b_{22}, a_{12}) & \cdots & \max(b_{2n}, a_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ \max(b_{n1}, a_{n1}) & \max(b_{n2}, a_{n2}) & \cdots & \max(b_{nn}, a_{nn}) \end{bmatrix}. \end{aligned}$$

Thus $A+B = B+A$. It follows that the addition of fuzzy matrices is commutative.

(2) Observe the following:

$$
\begin{aligned}
(A+B)+C &= \left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}\right) + \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \\
&= \begin{bmatrix} \max(a_{11}, b_{11}) & \max(a_{12}, b_{12}) & \cdots & \max(a_{1n}, b_{1n}) \\ \max(a_{21}, b_{21}) & \max(a_{22}, b_{12}) & \cdots & \max(a_{2n}, b_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ \max(a_{n1}, b_{n1}) & \max(a_{n2}, b_{n2}) & \cdots & \max(a_{nn}, b_{nn}) \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \\
&= \begin{bmatrix} \max(\max(a_{11}, b_{11}), c_{11}) & \max(\max(a_{12}, b_{12}), c_{12}) & \cdots & \max(\max(a_{1n}, b_{1n}), c_{1n}) \\ \max(\max(a_{21}, b_{21}), c_{21}) & \max(\max(a_{22}, b_{22}), c_{22}) & \cdots & \max(\max(a_{2n}, b_{2n}), c_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ \max(\max(a_{n1}, b_{n1}), c_{n1}) & \max(\max(a_{n2}, b_{n2}), c_{n2} & \cdots & \max(\max(a_{nn}, b_{nn}), c_{nn}) \end{bmatrix} \\
&= \begin{bmatrix} \max(a_{11}, b_{11}, c_{11}) & \max(a_{12}, b_{12}, c_{12}) & \cdots & \max(a_{1n}, b_{1n}, c_{1n}) \\ \max(a_{21}, b_{21}, c_{21}) & \max(a_{22}, b_{22}, c_{22}) & \cdots & \max(a_{2n}, b_{2n}, c_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ \max(a_{n1}, b_{n1}, c_{n1}) & \max(a_{n2}, b_{n2}, c_{n2} & \cdots & \max(a_{nn}, b_{nn}, c_{nn}) \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
A+(B+C) &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} + \left(\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}\right) \\
&= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} \max(b_{11}, c_{11}) & \max(b_{12}, c_{12}) & \cdots & \max(b_{1n}, c_{1n}) \\ \max(b_{21}, c_{21}) & \max(b_{22}, c_{12}) & \cdots & \max(b_{2n}, c_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ \max(b_{n1}, c_{n1}) & \max(b_{n2}, c_{n2}) & \cdots & \max(b_{nn}, c_{nn}) \end{bmatrix} \\
&= \begin{bmatrix} \max(a_{11}, \max(b_{11}, c_{11})) & \max(a_{12}, \max(b_{12}), c_{12})) & \cdots & \max(a_{1n}, \max(b_{1n}, c_{1n})) \\ \max(a_{21}, \max(b_{21}), c_{21})) & \max(a_{22}, \max(b_{22}, c_{22})) & \cdots & \max(a_{2n}, \max(b_{2n}, c_{2n})) \\ \vdots & \vdots & \ddots & \vdots \\ \max(a_{n1}, \max(b_{n1}, c_{n1})) & \max(a_{n2}, \max(b_{n2}, c_{n2})) & \cdots & \max(a_{nn}, \max(b_{nn}, c_{nn})) \end{bmatrix} \\
&= \begin{bmatrix} \max(a_{11}, b_{11}, c_{11}) & \max(a_{12}, b_{12}, c_{12}) & \cdots & \max(a_{1n}, b_{1n}, c_{1n}) \\ \max(a_{21}, b_{21}, c_{21}) & \max(a_{22}, b_{22}, c_{22}) & \cdots & \max(a_{2n}, b_{2n}, c_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ \max(a_{n1}, b_{n1}, c_{n1}) & \max(a_{n2}, b_{n2}, c_{n2}) & \cdots & \max(a_{nn}, b_{nn}, c_{nn}) \end{bmatrix}.
\end{aligned}
$$

Thus $(A+B)+C = A+(B+C)$. It follows that addition of fuzzy matrices is associative.

(3) Observe the following:

$$\begin{aligned} A+0 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\ &= \begin{bmatrix} \max(a_{11},0) & \max(a_{12},0) & \cdots & \max(a_{1n},0) \\ \max(a_{21},0) & \max(a_{22},0) & \cdots & \max(a_{2n},0) \\ \vdots & \vdots & \ddots & \vdots \\ \max(a_{n1},0) & \max(a_{n2},0) & \cdots & \max(a_{nn},0) \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}. \end{bmatrix} \end{aligned}$$

Thus $A+0=A$. Since fuzzy matrix addition is commutative (Property 1), it follows that $A+0 = 0+A = A$. □

**Proposition 2.3.** *Let $A$ be an $n \times n$ fuzzy matrix. With the* ***fuzzy subtraction*** *defined in Definition 2.1, we have the following:*

(1) $0 - A = 0$,

(2) $A - A = 0$,

(3) $A - 0 = A$,

*Proof.* Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$.

(1) Observe the following:

$$\begin{aligned}
0 - A &= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \\
&= \begin{bmatrix} 0 - a_{11} & 0 - a_{12} & \cdots & 0 - a_{1n} \\ 0 - a_{21} & 0 - a_{22} & \cdots & 0 - a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 - a_{n1} & 0 - a_{n2} & \cdots & 0 - a_{nn} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\
&= 0.
\end{aligned}$$

Thus $0 - A = 0$.

(2) Observe the following:

$$\begin{aligned}
A - A &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \\
&= \begin{bmatrix} a_{11} - a_{11} & a_{12} - a_{12} & \cdots & a_{1n} - a_{1n} \\ a_{21} - a_{21} & a_{22} - a_{22} & \cdots & a_{2n} - a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} - a_{n1} & a_{n2} - a_{n2} & \cdots & a_{nn} - a_{nn} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\
&= 0.
\end{aligned}$$

Thus $A - A = 0$.

(3) Let $i, j \in \mathbb{Z}^+$. Note that for any $1 \leq i, j \leq n$, $a_{ij} \geq 0$. Suppose $a_{ij} > 0$. Observe the

following:

$$
\begin{aligned}
A - 0 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\
&= \begin{bmatrix} a_{11} - 0 & a_{12} - 0 & \cdots & a_{1n} - 0 \\ a_{21} - 0 & a_{22} - 0 & \cdots & a_{2n} - 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} - 0 & a_{n2} - 0 & \cdots & a_{nn} - 0 \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \\
&= A.
\end{aligned}
$$

In other words, $a_{ij} - 0 = a_{ij}$ since $a_{ij} > 0$. Now suppose $a_{ij} = 0$. Then $a_{ij} - 0 = 0 = a_{ij}$ since $a_{ij} \leq 0$. Thus $A - 0 = A$. □

**Lemma 2.4.** *The* ***fuzzy multiplication*** *is distributive with respect to the fuzzy addition. In other words, if* $a, b, c \in [0, 1]$, *then* $a(b+c) = ab+ac$; *that is,* $\min(a, \max(b, c)) = \max(\min(a, b), \min(a, c))$.

*Proof.* Let $a, b, c \in [0, 1]$. It suffices to consider the following six cases:

(1) $a \leq b \leq c$,

(2) $a \leq c \leq b$,

(3) $b \leq a \leq c$,

(4) $b \leq c \leq a$,

(5) $c \leq a \leq b$,

(6) $c \leq b \leq a$.

For example, for case (1), we have:

$$
\begin{aligned}
a(b + c) &= \min(a, \max(b, c)) \\
&= \min(a, c) \\
&= a.
\end{aligned}
$$

On the other hand,

$$\begin{aligned} ab + ac &= \max(\min(a,b), \min(a,c)) \\ &= \max(a,a) \\ &= a. \end{aligned}$$

Thus case (1) is proved. The proofs of the other five cases are similar to case (1). ☐

**Proposition 2.5.** *Let $A, B, C$ be $n \times n$ fuzzy matrices. Then with the **fuzzy operations** we have:*

(1) $A \cdot 0 = 0 \cdot A = 0$,

(2) $A(B+C) = AB + AC$, *(Distributive)*

(3) $A \cdot I = I \cdot A = A$ *(Multiplicative Identity)*

(4) $(AB)C = A(BC)$ *(Associativity).*

*Proof.* Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$, and $C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$ be fuzzy matrices, and let $i, j \in \mathbb{Z}^+$. (1) Observe the following:

$$\begin{aligned} A \cdot 0 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\ &= \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}, \end{aligned}$$

where for each $1 \le i, j \le n$, $x_{ij} = \max\{\min(a_{ik}, 0), 1 \le k \le n\}$. Note that $a_{ik} \ge 0$, and therefore $\min(a_{ik}, 0) = 0$. It follows that $x_{ij} = 0$.

On the other hand, we have:

$$\begin{aligned} 0 \cdot A &= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \\ &= \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}, \end{aligned}$$

where for each $1 \leq i, j \leq n$, $x_{ij} = \max\{\min(0, a_{kj}), 1 \leq k \leq n\}$. Note that $a_{kj} \geq 0$, and therefore $\min(0, a_{kj}) = 0$. It follows that $x_{ij} = 0$. Thus, we have that $A \cdot 0 = 0 \cdot A = 0$.

(2) Let $1 \leq i, j \leq n$. Note that $B + C = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix}$, where $v_{ij} = \max(b_{ij}, c_{ij})$. Then $A(B+C) = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{bmatrix}$, where $w_{ij} = \max\{\min(a_{ik}, v_{kj}),\ 1 \leq k \leq n\}$. Now note that $AB = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}$, where $x_{ij} = \max\{\min(a_{ik}, b_{kj}),\ 1 \leq k \leq n\}$, and $AC = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix}$, where $y_{ij} = \max\{\min(a_{ik}, c_{kj}),\ 1 \leq k \leq n\}$. It follows that $AB + AC = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{bmatrix}$, where $z_{ij} = \max\{\min(x_{ij}, y_{ij}), 1 \leq k \leq n\}$. In other words, for any $1 \leq k \leq n$, we have:

$$z_{ij} = \max\{\max(\min(a_{ik}, b_{kj})), \max(\min(a_{ik}, c_{kj}))\}.$$

Note that $w_{ij} = \max\{min(a_{ik}, v_{kj})\} = \max\{\min(a_{ik}, \max(b_{kj}, c_{kj})), 1 \leq k \leq n\}$. It suffices to show that $w_{ij} = z_{ij}$; that is, $\max\{\min(x_{ij}, y_{ij}), 1 \leq k \leq n\} = \max\{\min(a_{ik}, \max(b_{kj}, c_{kj})), 1 \leq k \leq n\}$. Observe the following:

$$\begin{aligned}
z_{ij} &= \max\{\max(\min(a_{ik}, b_{kj})), \max(\min(a_{ik}, c_{kj}))\} \\
&= (a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{ki}) + (a_{i1}c_{1j} + a_{i2}c_{2j} + \cdots + a_{ik}c_{kj}) \\
&= (a_{i1}b_{1j} + a_{i1}c_{1j}) + (a_{i2}b_{2j} + a_{i2}c_{2j}) + \cdots + (a_{ik}b_{kj} + a_{ik}c_{kj}) \\
&= \max\{\min(a_{ik}, b_{kj}), \min(a_{ik}, c_{kj})\}
\end{aligned}$$

By Lemma 2.4, we have that $\max\{\min(a_{ik}, b_{kj}), \min(a_{ik}, c_{kj})\} = \min\{a_{ik}, \max(b_{kj}, c_{kj})\}$. Now observe the following:

$$\begin{aligned}
z_{ij} &= \min\{a_{ik}, \max(b_{kj}, c_{kj})\} \\
&= \min(a_{ik}, v_{kj}) \\
&= w_{ij}.
\end{aligned}$$

Thus, $z_{ij} = w_{ij}$. It follows that $A(B+C) = AB + AC$.

(3) Let $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$. Observe the following:

$$\begin{aligned}
A \cdot I &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \\
&= \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix},
\end{aligned}$$

where for each $1 \leq i \leq n$ and $1 \leq j \leq n$, $x_{ij} = \max\{\min(a_{ij}, 1), \min(a_{pq}, 0), 1 \leq p \leq n, 1 \leq q \leq n, p \neq i, q \neq j\}$. It follows that $x_{ij} = \max(a_{ij}, 0) = a_{ij}$. Therefore $A \cdot I = A$.

On the other hand,

$$
\begin{aligned}
I \cdot A &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \\
&= \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix},
\end{aligned}
$$

where for each $1 \leq i \leq n$ and $1 \leq j \leq n$, $x_{ij} = \max\{\min(1, a_{ij}), \min(0, a_{pq}), 1 \leq p \leq n, 1 \leq q \leq n, p \neq i, q \neq j\}$. It follows that $x_{ij} = \max(0, a_{ij}) = a_{ij}$. Thus, $A \cdot I = I \cdot A = A$.

(4) Observe the following:

$$
\begin{aligned}
(AB)C &= \sum_{k=1}^{n} (AB)_{ik} c_{kj} \\
&= \sum_{k=1}^{n} (\sum_{m=1}^{n} a_{im} b_{mk}) c_{kj} \\
&= \sum_{k,m=1}^{n} a_{im} b_{mk} c_{kj} \\
&= \sum_{k,m=1}^{n} \min(a_{im}, b_{mk}, c_{kj}).
\end{aligned}
$$

Similarly, observe the following:

$$
\begin{aligned}
A(BC) &= \sum_{k=1}^{n} a_{ik} (BC)_{kj} \\
&= \sum_{k=1}^{n} a_{ik} (\sum_{m=1}^{n} b_{km} c_{mj}) \\
&= \sum_{k,m=1}^{n} a_{ik} b_{km} c_{mj} \\
&= \sum_{k,m=1}^{n} \min(a_{ik}, b_{km}, c_{mj}) \\
&= \sum_{k,m=1}^{n} \min(a_{im}, b_{mk}, c_{kj}) \\
&= (AB)C.
\end{aligned}
$$

Thus $(AB)C = A(BC)$. □

## 3. Determinants of Square Fuzzy Matrices

**Definition 3.1.** *The determinant $|A|$ of an $n \times n$ fuzzy matrix $A$ is defined as*

$$|A| = \det(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)},$$

*where $S_n$ denotes the symmetric group of all permutations of the indices $(1, 2, ..., n)$. [2]*

**Example 3.2.** *Let* $A = \begin{bmatrix} 0.7 & 0.1 & 0.9 \\ 0 & 0.4 & 1 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$. *Then*

$$\begin{aligned}
\det(A) &= 0.7 \begin{vmatrix} 0.4 & 1 \\ 0.3 & 0.5 \end{vmatrix} + 0.1 \begin{vmatrix} 0 & 1 \\ 0.2 & 0.5 \end{vmatrix} + 0.9 \begin{vmatrix} 0 & 0.4 \\ 0.2 & 0.3 \end{vmatrix} \\
&= 0.7(\min(0.4, 0.5) + \min(1, 0.3)) + 0.1(\min(0, 0.5) + \min(1, 0.2)) + 0.9(\min(0, 0.3) + \min(0.4, 0.2)) \\
&= 0.7(0.4 + 0.3) + 0.1(0 + 0.2) + 0.9(0 + 0.2) \\
&= 0.7(0.4) + 0.1(0.2) + 0.9(0.2) \\
&= 0.4 + 0.1 + 0.2 \\
&= 0.4.
\end{aligned}$$

**Remark 3.3.**
*(1) Recall that in the case of classical matrices, we alternate between addition and subtraction when calculating the determinant, but in the case of fuzzy matrices, we only use* ***fuzzy addition****.*
*(2) We have that $\det(A)\det(B) = \det(AB)$. But this is not always true for fuzzy matrices. For instance, consider the following examples:*

**Example 3.4.** *Let* $A = \begin{bmatrix} 0.7 & 0.1 & 0.9 \\ 0 & 0.4 & 1 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.8 & 1 & 0.3 \\ 0.4 & 0.9 & 0.4 \end{bmatrix}$ *be two fuzzy matrices. Then*

$$\begin{aligned}
\det(A) &= 0.7 \begin{vmatrix} 0.4 & 1 \\ 0.3 & 0.5 \end{vmatrix} + 0.1 \begin{vmatrix} 0 & 1 \\ 0.2 & 0.5 \end{vmatrix} + 0.9 \begin{vmatrix} 0 & 0.4 \\ 0.2 & 0.3 \end{vmatrix} \\
&= 0.7(\min(0.4, 0.5) + \min(1, 0.3)) + 0.1(\min(0, 0.5) + \min(1, 0.2)) + 0.9(\min(0, 0.3) + \min(0.4, 0.2)) \\
&= 0.7(0.4 + 0.3) + 0.1(0 + 0.2) + 0.9(0 + 0.2) \\
&= 0.7(0.4) + 0.1(0.2) + 0.9(0.2) \\
&= 0.4 + 0.1 + 0.2 \\
&= 0.4.
\end{aligned}$$

*Now observe that*

$$\begin{aligned}
\det(B) &= 0.2\begin{vmatrix} 1 & 0.3 \\ 0.9 & 0.4 \end{vmatrix} + 0.1\begin{vmatrix} 0.8 & 0.3 \\ 0.4 & 0.4 \end{vmatrix} + 0\begin{vmatrix} 0.8 & 1 \\ 0.4 & 0.9 \end{vmatrix} \\
&= 0.2(\min(1, 0.4) + \min(0.3, 0.9)) + 0.1(\min(0.8, 0.4) + \min(0.3, 0.4)) + 0(\min(0.8, 0.9) + \min(1, 0.4)) \\
&= 0.2(0.4 + 0.3) + 0.1(0.4 + 0.3) + 0(0.8 + 0.4) \\
&= 0.2(0.4) + 0.1(0.4) + 0(0.8) \\
&= 0.2 + 0.1 + 0 \\
&= 0.4.
\end{aligned}$$

*It follows that* $\det(A) \cdot \det(B) = \min(0.4, 0.2) = 0.2$. *On the other hand;*

$$\begin{aligned}
AB &= \begin{bmatrix} 0.7 & 0.1 & 0.9 \\ 0 & 0.4 & 1 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.8 & 1 & 0.3 \\ 0.4 & 0.9 & 0.4 \end{bmatrix} \\
&= \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \\
&= \begin{bmatrix} \max(0.2, 0.1, 0.4) & \max(0.1, 0.1, 0.9) & \max(0, 0.1, 0.4) \\ \max(0, 0.4, 0.4) & \max(0, 0.4, 0.9) & \max(0, 0.3, 0.4) \\ \max(0.2, 0.3, 0.5) & \max(0.1, 0.3, 0.5) & \max(0, 0.3, 0.4) \end{bmatrix} \\
&= \begin{bmatrix} 0.4 & 0.9 & 0.4 \\ 0.4 & 0.9 & 0.4 \\ 0.4 & 0.5 & 0.4 \end{bmatrix},
\end{aligned}$$

*where*

$$\begin{aligned}
x_{11} &= \max(\min(0.7, 0.2), \min(0.1, 0.8), \min(0.9, 0.4)), \\
x_{12} &= \max(\min(0.7, 0.1), \min(0.1, 1), \min(0.9, 0.9)), \\
x_{13} &= \max(\min(0.7, 0), \min(0.1, 0.3), \min(0.9, 0.4)), \\
x_{21} &= \max(\min(0, 0.2), \min(0.4, 0.8), \min(1, 0.4)), \\
x_{22} &= \max(\min(0, 0.1), \min(0.4, 1), \min(1, 0.9)), \\
x_{23} &= \max(\min(0, 0), \min(0.4, 0.3), \min(1, 0.4)), \\
x_{31} &= \max(\min(0.2, 0.2), \min(0.3, 0.8), \min(0.5, 0.4)), \\
x_{32} &= \max(\min(0.2, 0.1), \min(0.3, 1), \min(0.5, 0.9)), \\
x_{33} &= \max(\min(0.2, 0), \min(0.3, 0.3), \min(0.5, 0.4)).
\end{aligned}$$

*Then*

$$\begin{aligned}
\det(AB) &= 0.4\begin{vmatrix}0.9 & 0.4\\ 0.5 & 0.4\end{vmatrix} + 0.9\begin{vmatrix}0.4 & 0.4\\ 0.4 & 0.4\end{vmatrix} + 0.4\begin{vmatrix}0.4 & 0.9\\ 0.4 & 0.5\end{vmatrix}\\
&= 0.4(\min(0.9, 0.4) + \min(0.4, 0.5)) + 0.9(\min(0.4, 0.4) + \min(0.4, 0.4)) + 0.4(\min(0.4, 0.5) + \min(0.9, 0\\
&= 0.4(0.4 + 0.4) + 0.9(0.4 + 0.4) + 0.4(0.4 + 0.4)\\
&= 0.4(0.4) + 0.9(0.4) + 0.4(0.4)\\
&= 0.4 + 0.4 + 0.4\\
&= 0.4.
\end{aligned}$$

*Therefore* $\det(A)\det(B) = 0.2 \neq 0.4 = \det(AB)$. *Also, note that* $\det(A) + \det(B) = 0.4$.

*Now observe the following:*

$$\begin{aligned}
A + B &= \begin{bmatrix}0.7 & 0.1 & 0.9\\ 0 & 0.4 & 1\\ 0.2 & 0.3 & 0.5\end{bmatrix} + \begin{bmatrix}0.2 & 0.1 & 0\\ 0.8 & 1 & 0.3\\ 0.4 & 0.9 & 0.4\end{bmatrix}\\
&= \begin{bmatrix}\max(0.7, 0.2) & \max(0.1, 0.1) & \max(0.9, 0)\\ \max(0, 0.8) & \max(0.4, 1) & \max(1, 0.3)\\ \max(0.2, 0.4) & \max(0.3, 0.9) & \max(0.5, 0.4)\end{bmatrix}\\
&= \begin{bmatrix}0.7 & 0.1 & 0.9\\ 0.8 & 1 & 1\\ 0.4 & 0.9 & 0.5\end{bmatrix}.
\end{aligned}$$

$$\begin{aligned}
\det(A + B) &= 0.7\begin{vmatrix}1 & 1\\ 0.9 & 0.5\end{vmatrix} + 0.1\begin{vmatrix}0.8 & 1\\ 0.4 & 0.5\end{vmatrix} + 0.9\begin{vmatrix}0.8 & 1\\ 0.4 & 0.9\end{vmatrix}\\
&= 0.7(\min(1, 0.5) + \min(1, 0.9)) + 0.1(\min(0.8, 0.5) + \min(1, 0.4)) + 0.9(\min(0.8, 0.9) + \min(1, 0.4))\\
&= 0.7(0.5 + 0.9) + 0.1(0.5 + 0.4) + 0.9(0.8 + 0.4)\\
&= 0.7(0.9) + 0.1(0.5) + 0.9(0.8)\\
&= 0.7 + 0.1 + 0.8\\
&= 0.8.
\end{aligned}$$

*Therefore* $\det(A) + \det(B) = 0.4 \neq 0.8 = \det(A + B)$. *Thus* $\det(A) + \det(B) \neq \det(A + B)$.

**Proposition 3.5.** *Let* $A = \begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix}$ *be an* $n \times n$ *fuzzy matrix. Then* $\det(A) = \det(A^T)$.

*Proof.* Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. It follows that $\det(A) = \max(\min(a_{11}a_{22}), \min(a_{12}a_{21}))$. Now note that $A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$. Then $\det(A^T) = \max(\min(a_{11}a_{22}), \min(a_{12}a_{21})) = \det(A)$. Now let $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$. Then $B^T = \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{bmatrix}$. Observe the following:

$$\det(A) = b_{11}\begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix} + b_{21}\begin{vmatrix} b_{12} & b_{13} \\ b_{32} & b_{33} \end{vmatrix} + b_{31}\begin{vmatrix} b_{12} & b_{13} \\ b_{22} & b_{23} \end{vmatrix}$$

Since for any $2 \times 2$ fuzzy matrix $A$, we have that $\det(A) = \det(A^T)$, observe the following:

$$\begin{aligned}
\det(B) &= b_{11}\begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix} + b_{21}\begin{vmatrix} b_{12} & b_{13} \\ b_{32} & b_{33} \end{vmatrix} + b_{31}\begin{vmatrix} b_{12} & b_{13} \\ b_{22} & b_{23} \end{vmatrix} \\
&= b_{11}\begin{vmatrix} b_{22} & b_{32} \\ b_{23} & b_{33} \end{vmatrix} + b_{21}\begin{vmatrix} b_{12} & b_{32} \\ b_{13} & b_{33} \end{vmatrix} + b_{31}\begin{vmatrix} b_{12} & b_{22} \\ b_{13} & b_{23} \end{vmatrix} \\
&= \begin{vmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{33} \\ b_{13} & b_{23} & b_{33} \end{vmatrix} \\
&= \det(B^T).
\end{aligned}$$

Thus, by iteration, we deduce that $\det(A) = \det(A^T)$ for any $n \times n$ fuzzy matrix $A$. □

## 4. Traces of Fuzzy Matrices

**Proposition 4.1.** *Let $A$ and $B$ be two $n \times n$ fuzzy matrices, and let $\lambda$ be a real number such that $\lambda \in [0, 1]$. Then we have the following:*

(1) $\mathrm{Tr}(A) + \mathrm{Tr}(B) = \mathrm{Tr}(A + B)$,

(2) $\mathrm{Tr}(A) = \mathrm{Tr}(A^T)$,

(3) $\mathrm{Tr}(\lambda A) = \lambda\,\mathrm{Tr}(A)$.

*Proof.* Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$ be two fuzzy matrices.

(1) Note that $\mathrm{Tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \max(a_{11}, a_{22}, ..., a_{nn})$ and $\mathrm{Tr}(B) = b_{11} + b_{22} + \cdots b_{nn} =$

$\max(a_{11}, a_{22}, ..., a_{nn})$. Then we obtain:

$$\begin{aligned} \mathrm{Tr}(A) + \mathrm{Tr}(B) &= \max(\max(a_{11}, a_{22}, ..., a_{nn}), \max(b_{11}, b_{22}, ..., b_{nn})) \\ &= \max(a_{11}, a_{22}, ..., a_{nn}, b_{11}, b_{22}, ..., b_{nn}). \end{aligned}$$

$$A + B = \begin{bmatrix} \max(a_{11}, b_{11}) & \max(a_{12}, b_{12}) & \cdots & \max(a_{1n}, b_{1n}) \\ \max(a_{21}, b_{21}) & \max(a_{22}, b_{22}) & \cdots & \max(a_{2n}, b_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ \max(a_{n1}, b_{n1}) & \max(a_{n2}, b_{n2}) & \cdots & \max(a_{nn}, b_{nn}) \end{bmatrix}.$$

It follows that

$$\begin{aligned} \mathrm{Tr}(A + B) &= \max(\max(a_{11}, b_{11}), \max(a_{22}, b_{22}), ..., \max(a_{nn}, b_{nn})) \\ &= \max(a_{11}, b_{11}, a_{22}, b_{22}, ..., a_{nn}, b_{nn}) \\ &= \max(a_{11}, a_{22}, ..., a_{nn}, b_{11}, b_{22}, ..., b_{nn}). \end{aligned}$$

Therefore $\mathrm{Tr}(A) + \mathrm{Tr}(B) = \mathrm{Tr}(A + B)$.

(2) Note that $A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$. Thus $\mathrm{Tr}(A^T) = \max(a_{11}, a_{22}, ..., a_{nn}) = \mathrm{Tr}(A)$.

(3) Let $A$ be an $n \times n$ fuzzy matrix, and let $\lambda$ be a real number in the interval $[0, 1]$. Then $\lambda A = \{\min(\lambda, a_{ij}) : 1 \leq i \leq n, 1 \leq j \leq n\}$. It follows that $\mathrm{Tr}(\lambda A) = \max\{\min(\lambda, a_{ij})\}$. Note that $\mathrm{Tr}(A) = \max\{a_{ii}\}$. Then $\lambda \,\mathrm{Tr}(A) = \min\{\lambda, \max(a_{ii})\}$. Then by Lemma 2.4, $\mathrm{Tr}(\lambda A) = \lambda \,\mathrm{Tr}(A)$. □

## 5. *K*−Idempotent Fuzzy Matrices.

**Definition 5.1.** *A fuzzy matrix* $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ *is said to be* $K-$*idempotent if, with respect to fuzzy operative, we have* $KA^2K = A$, *where* $K = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$. *[5]*

**Example 5.2.** *Let* $A = \begin{bmatrix} 1 & 0.1 & 0.1 \\ 0.4 & 1 & 0.2 \\ 0.7 & 0.6 & 0.8 \end{bmatrix}$ *and* $K = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. *Then, with respect to fuzzy operative,* $KA^2K = \begin{bmatrix} 1 & 0.1 & 0.1 \\ 0.4 & 1 & 0.2 \\ 0.7 & 0.6 & 0.8 \end{bmatrix}$. *Hence $A$ is a $K-$idempotent fuzzy matrix.*

**Proposition 5.3.** *(Characterization of a $2 \times 2$ $K-$idempotent fuzzy matrix)* *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a $2 \times 2$ $K-$idempotent fuzzy matrix. Then*

$$\begin{aligned} a &= d \\ b &= c. \end{aligned}$$

*Proof.* Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a $2 \times 2$ $K-$idempotent matrix. We want $KA^2K = A$, where $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Observe the following:

$$\begin{aligned} A^2 &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} \max(\min(a,a), \min(b,c)) & \max(\min(a,b), \min(b,d)) \\ \max(\min(c,a), \min(d,c)) & \max(\min(c,b), \min(d,d)) \end{bmatrix} \\ &= \begin{bmatrix} \max(a, \min(b,c)) & \max(\min(a,b), \min(b,d)) \\ \max(\min(c,a), \min(d,c)) & \max(\min(c,b), d) \end{bmatrix} \end{aligned}$$

Now we will compute $KA^2$ :

$$\begin{aligned} KA^2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \max(a, \min(b,c)) & \max(\min(a,b), \min(b,d)) \\ \max(\min(c,a), \min(d,c)) & \max(\min(c,b), d) \end{bmatrix} \\ &= \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \\ &= \begin{bmatrix} \max(0, \max(\min(c,a))) & \max(0, \max(\min(c,b), d)) \\ \max(\max(a, \min(b,c)), 0) & \max(\max(\min(a,b), \min(b,d)), 0) \end{bmatrix} \\ &= \begin{bmatrix} \max(\min(c,a), \min(d,c)) & \max(\min(c,b), d) \\ \max(a, \min(b,c)) & \max(\min(a,b), \min(b,d)) \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned}
x_{11} &= \max(\min(0, \max(a, \min(b, c))), \min(1, \max(\min(c, a), \min(d, c)))), \\
x_{12} &= \max(\min(0, \max(\min(a, b), \min(b, d))), \min(1, \max(\min(c, b), d))), \\
x_{21} &= \max(\min(1, \max(a, \min(b, c))), \min(0, \max(\min(c, a), \min(d, c)))), \\
x_{22} &= \max(\min(1, \max(\min(a, b), \min(b, d))), \min(0, \max(\min(c, b), d))).
\end{aligned}$$

Now we will compute $KA^2K$ :

$$\begin{aligned}
KA^2K &= \begin{bmatrix} \max(\min(c, a), \min(d, c)) & \max(\min(c, b), d) \\ \max(a, \min(b, c)) & \max(\min(a, b), \min(b, d)) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \\
&= \begin{bmatrix} \max(0, \max(\min(c, b), d)) & \max(\max(\min(c, a), \min(d, c), 0)), \\ \max(0, \max(\min(a, b), \min(b, d))) & \max(\max(\max(a, \min(b, c)), 0)) \end{bmatrix} \\
&= \begin{bmatrix} \max(\min(c, b), d) & \max(\min(c, a), \min(d, c)) \\ \max(\min(a, b), \min(b, d)) & \max(a, \min(b, c)) \end{bmatrix},
\end{aligned}$$

where

$$\begin{aligned}
y_{11} &= \max(\min(\max(\min(c, a), 0)), \min(\max(\min(c, b), d), 1)) \\
y_{12} &= \max(\min(\max(\min(c, a), 1)), \min(\max(\min(c, b), d))), \\
y_{21} &= \max(\min(\max(\max(a, \min(b, c))), 0), \min(\max(\min(a, b), \min(b, d)), 1)) \\
y_{22} &= \max(\min(\max(\max(a, \min(b, c)), 1)), \min(\max(a, b), \min(b, d), 0)).
\end{aligned}$$

Thus, we obtain the following:

$$\begin{aligned}
a &= \max(\min(c, b), d), \\
b &= \max(\min(c, a), \min(d, c)), \\
c &= \max(\min(a, b), \min(b, d)), \\
d &= \max(a, \min(b, c)).
\end{aligned}$$

Since $a = \max(\min(c, b), d)$, it follows that $a \geq d$. Similarly, $d = \max(a, \min(b, c))$ implies that $d \geq a$. Since $a \geq d$ and $d \geq a$, it follows that $a = d$. Substituting $d$ for $a$, observe the following:

$$\begin{aligned}
b &= \max(\min(c, a), \min(d, c)) \\
&= \max(\min(c, a), \min(a, c)) \\
&= \min(c, a) \\
&\leq c.
\end{aligned}$$

Similarly,

$$\begin{aligned} c &= \max(\min(a,b), \min(b,d)) \\ &= \max(\min(a,b), \min(b,a)) \\ &= \min(a,b) \\ &\leq b. \end{aligned}$$

Since $b \leq c$ and $c \leq b$, it follows that $c = b$. Therefore the general form of a $2 \times 2$ $K-$idempotent fuzzy matrix is $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$. □

**Remark 5.4.** *Let* $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ *and* $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$ *be two* $K-$*idempotent fuzzy matrices. In their paper [5], Muthugurupackjam and Krishnamohan stated that the following are true:*

*(1)* $\det(A)\det(B) = \det(AB)$,

*(2)* $\det(A) + \det(B) = \det(A+B)$.

*But a proof is not provided. In the next result, we will prove the case of a* $2 \times 2$ $K-$*idempotent matrix.*

**Proposition 5.5.** *Let* $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ *and* $B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$ *be two* $K-$*idemptotent fuzzy matrices. Then* $\det(A+B) = \det(A) + \det(B)$ *and* $\det(A)\det(B) = \det(AB)$.

*Proof.* Let $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$ be two $K-$idemptotent fuzzy matrices. Then $A + B = \begin{bmatrix} a+c & b+d \\ b+d & a+c \end{bmatrix}$. The determinant of $A$ is given by the following:

$$\begin{aligned} \det(A) &= \max(\min(a,a), \min(b,b)) \\ &= \max(a,b) \\ &= a+b. \end{aligned}$$

Similarly, the determinant of $B$ is given by the following:

$$\begin{aligned} \det(B) &= \max(\min(c,c), \min(d,d)) \\ &= \max(c,d) \\ &= c+d. \end{aligned}$$

Thus $\det(A)+\det(B)=a+b+c+d$. Now observe the following:

$$\begin{aligned}\det(A+B) &= \max(\min(a+c,a+c),\min(b+d,b+d))\\ &= \max(a+c,b+d)\\ &= (a+c)+(b+d)\\ &= a+b+c+d\\ &= \det(A)+\det(B).\end{aligned}$$

Thus $\det(A)+\det(B)=\det(A+B)$.

Now we want to show that $\det(A)\det(B)=\det(AB)$. Note that $AB=\begin{bmatrix} ac+bd & ad+bc \\ ad+bc & ac+bd \end{bmatrix}$. Note that $\det(A)\det(B)=(a+b)(c+d)=\min(a+b,c+d)$. Since the distributive property for fuzzy numbers holds, observe the following:

$$\begin{aligned}\det(A)\det(B) &= \min(a+b,c+d)\\ &= \max(\min(a,c),\min(a,d),\min(b,c),\min(b,d))\\ &= ac+bd+ad+bc.\end{aligned}$$

$$\begin{aligned}\det(AB) &= (ac+bd)^2+(ad+bc)^2\\ &= \max(\min(ac+bd,ac+bd),\min(ad+bc,ad+bc))\\ &= ac+bd+ad+bc\\ &= \det(A)\det(B)\end{aligned}$$

Thus $\det(A)\det(B)=\det(AB)$. □

## 6. Triangular Fuzzy Matrices

**Proposition 6.1.** *Let* $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ *be an* $n\times n$ *be an upper triangular fuzzy matrix. Then* $\det(A)=a_{11}a_{22}a_{33}\cdots a_{nn}$.

*Proof.* Let $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ be an $n\times n$ be an upper triangular fuzzy matrix. Observe the following:

$$
\begin{aligned}
\det(A) &= \max\left(\min\left(a_{11}, \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ 0 & 0 & \cdots & a_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}\right), 0\right) \\
&= \min\left(a_{11}, \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ 0 & 0 & \cdots & a_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}\right) \\
&= \min\left(a_{11}, \max\left(\min(a_{22}, \begin{vmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & a_{44} & \cdots & a_{4n} \\ 0 & 0 & \cdots & a_{5n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}\right), 0\right) \\
&= \min\left(a_{11}, \min\left(a_{22}, \begin{vmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & a_{44} & \cdots & a_{4n} \\ 0 & 0 & \cdots & a_{5n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}\right)\right) \\
&= \min\left(a_{11}, a_{22}, \begin{vmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & a_{44} & \cdots & a_{4n} \\ 0 & 0 & \cdots & a_{5n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}\right) \\
&= \min\left(a_{11}, a_{22}, a_{33}, ..., a_{nn}\right).
\end{aligned}
$$

□

**Proposition 6.2.** *(Characterization of a $K-$idempotent Triangular $n \times n$ Fuzzy Matrix)* *Let $A$ be a $K-$idempotent triangular $n \times n$ fuzzy matrix. Then for each $i, j \in \mathbb{Z}$ such that $1 \leq i, j \leq n$ and $i \neq j$, $a_{ij} = 0$. Furthermore,*

*(1) If $n$ is even, then $a_{ii} = a_{(n+1-i)(n+1-i)}$*

*(2) If $n$ is odd, then we have the following:*

$$\begin{aligned} a_{11} &= a_{nn} \\ a_{22} &= a_{(n-1)(n-1)} \\ &\vdots \\ a_{(\frac{n-1}{2})(\frac{n-1}{2})} &= a_{(\frac{n+3}{2})(\frac{n+3}{2})}. \end{aligned}$$

*Proof.* Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ be a fuzzy $n \times n$ upper triangular matrix. We want to show that $A = KA^2K$, where $K = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$. Observe the following:

$$\begin{aligned}
A^2 &= \begin{bmatrix} a_{11}^2 & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & a_{22}^2 & b_{23} & \cdots & b_{2n} \\ 0 & 0 & a_{33}^2 & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^2 \end{bmatrix} \\
KA^2 &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_{11}^2 & b_{12} & \cdots & b_{1n} \\ 0 & a_{22}^2 & \cdots & b_{2n} \\ 0 & 0 & \cdots & b_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & \cdots & 0 & a_{nn}^2 \\ 0 & 0 & \cdots & a_{(n-1)(n-1)}^2 & b_{(n-1)(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{22}^2 & \cdots & b_{2(n-1)} & b_{2n} \\ a_{11}^2 & b_{12} & \cdots & b_{1(n-1)} & b_{1n} \end{bmatrix} \\
KA^2K &= \begin{bmatrix} 0 & 0 & \cdots & 0 & a_{nn}^2 \\ 0 & 0 & \cdots & a_{(n-1)(n-1)}^2 & b_{(n-1)(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{22}^2 & \cdots & b_{2(n-1)} & b_{2n} \\ a_{11}^2 & b_{12} & \cdots & b_{1(n-1)} & b_{1n} \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} a_{nn}^2 & 0 & \cdots & 0 & 0 \\ b_{(n-1)(n)} & a_{(n-1)(n-1)}^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{2n} & b_{2(n-1)} & \cdots & a_{22}^2 & 0 \\ b_{1n} & b_{1(n-1)} & \cdots & b_{12} & a_{11}^2 \end{bmatrix}.
\end{aligned}$$

Recall that for any fuzzy number $a$, $a^2 = \min(a, a) = a$. Therefore we obtain the following:

$$\begin{bmatrix} a_{nn} & 0 & \cdots & 0 & 0 \\ b_{(n-1)(n)} & a_{(n-1)(n-1)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{2n} & b_{2(n-1)} & \cdots & a_{22} & 0 \\ b_{1n} & b_{1(n-1)} & \cdots & b_{12} & a_{11} \end{bmatrix}$$

Now we want to show that $A = KA^2K$; that is,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{nn} & 0 & \cdots & 0 & 0 \\ b_{(n-1)(n)} & a^2_{(n-1)(n-1)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{2n} & b_{2(n-1)} & \cdots & a_{22} & 0 \\ b_{1n} & b_{1(n-1)} & \cdots & b_{12} & a_{11} \end{bmatrix}.$$

From the above matrices, we deduce that for each $i, j \in \mathbb{N}$ where $1 \leq i \leq n-1$ and $2 \leq j \leq n$ such that $i \neq j$, $b_{ij} = 0$ and $a_{ij} = 0$. Furthermore, we deduce that if $n$ is even, then for each $1 \leq i \leq n$, $a_{ii} = a_{(n+1-i)(n+1-i)}$. If $n$ is odd, then we obtain the following:

$$\begin{aligned} a_{11} &= a_{nn} \\ a_{22} &= a_{(n-1)(n-1)} \\ &\vdots \\ a_{(\frac{n-1}{2})(\frac{n-1}{2})} &= a_{(\frac{n+3}{2})(\frac{n+3}{2})}. \end{aligned}$$

□


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