A Fractal Function Related to the Rudin-Shapiro Sequence

Bennett Haller

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Abstract

A fractal is a set that has some form of self-similarity and is seen ubiquitously in nature due to its ability to appear approximately the same at different levels. In this project, a fractal function Psi which is related to the Rudin-Shapiro sequence, is generated and studied. Two associated functions Delta and Phi are also generated as they are used to form the function Psi.

1 Introduction

A fractal is a geometric figure where part of the object has the same statistical characteristics as its whole figure and contains never-ending patterns. Fractal geometry is an extension of classical geometry (see [3]). It is used to make exact models of physical structures. As the scale changes, the complexity of the object remains the same. In other words, viewing a fractal object at microscopic levels will not change its complexity. Fractals can be seen throughout our everyday lives (see [3]). Between mountains, snowflakes or even DNA, fractal geometry is present within these objects. Fractals vary in the form which they are presented. The fractals which are self-similar are the most simple to describe. An object with self-similarity means that it is similar to a part of itself; each part of the object looks like the object as a whole. Using the Rudin-Shapiro sequence, all of the functions ψ, Δ, ϕ and ψ are formed. All of these functions will be analyzed in this paper.

2 Fractal Properties and Examples

2.1 Definition of a Fractal

A fractal F is defined by having the following characteristics in [1]:

(i) F has a fine structure, i.e. detail on arbitrarily small scales.

(ii) F is too irregular to be described in traditional geometrical language, both locally and globally.

(iii) Often F has some form of self-similarity, perhaps approximate or statistical

(iv) Usually, the 'fractal dimension' of F (defined in some way) is greater than its topological dimension.

(v) In most cases of interest F is defined in a very simple way, perhaps recursively.

2.2 Fractal Dimension

Typically a line is measured in first dimension 1, a plane is measured in the dimension 2 and a cube is measured in the dimension 3. Most fractal dimensions are not expressed by an integer. Typically the fractal dimension is greater than its topological dimension. There are two types of dimensions that are most commonly used with fractal geometry. The Hausdorff dimension is the most rigorous mathematical definition. The other is the box-counting dimension. The box-counting dimension is a way of determining the fractal dimension of a set S in a Euclidean space R^n or a metric space (X, d). Imagine a fractal S is lying on an evenly spaced grid, where the number of boxes required to cover the set is counted. The box-counting dimension can then be calculated to see how this number would change as the grid is made finer. There are several definitions of Box dimension[1] which are equivalent. For a set in a plane, we will chose the following:

Let $\delta > 0$ and consider a collection of squares in the δ-mesh.

In other words, the squares of the form $[m_1\delta, (m_1 + 1)\delta]$ x $[m_2\delta, (m_2 + 1)\delta](m_1, m_2 \in \mathbb{Z}$

Take F to be a non-empty bounded subset of $\mathbb{R}^2$ and let $N_\delta(F)$ be the number of δ-mesh squares intersecting F. Then an upper and lower Box dimension can be formed as $\overline{dim_B}(F) = \overline{\lim}_{\delta\to 0} \frac{logN_\delta(F)}{-log(\delta)}$ and $\underline{dim_B}(F) = \underline{lim}_{\delta\to 0} \frac{logN_\delta(F)}{-log(\delta)}$ respectively. If $\underline{dim_B}(F) = \overline{dim_B}(F)$ then the common value can be referred to as the Box dimension of F and written as $dim_B(F)$. In general, for $\delta_k = c^k (0 < c < 1)$ it is enough to take the limits as δ tends to 0, when δ_k tends to 0. Then,

$$dim_B F = lim_{\delta \to 0} \frac{log N_\delta F}{-log(\delta)} = lim_{k \to \infty} \frac{log N_{\delta_k} F}{-log(\delta)_k}$$

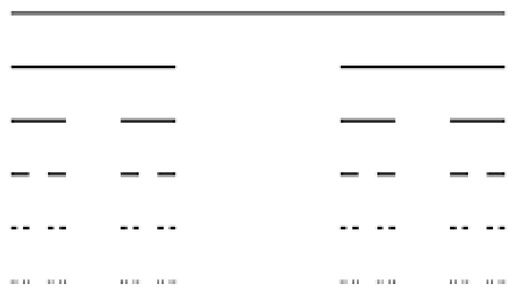
For many fractals which occur in nature, the Hausdorff and box-counting dimension coincide with one another. Once a fractal dimension is found, the measure can then be calculated in that dimension.

2.3 Fractal Measure

Once the appropriate dimension is found, the measure of the fractal can then be calculated. The Lebesgue measure is a standard way of assigning a measure to subsets of n-dimensional Euclidean space. In most cases, fractals cannot be defined in a n-Dimension. For example, Mandelbrot set out to find the measure of the coastline of Britain. As the coastline was analyzed at smaller levels, it became clear that the length was ∞ when measured in the first dimension. If one wishes to compare different coastlines from the viewpoint of their "extent," length is an inadequate concept (see [3]). Mandelbrot then was able to formulate an equation to calculate a fractal dimension of 1.24. This allowed the coastline to be appropriately measured. The Hausdorff measure is a generalization of the Lebesgue measure that can be applied to fractal sets. Its measure is a type of outer measure which assigns a number $[0, \infty]$ to each set in $\mathbb{R}^n$ (any metric space).

2.4 The Cantor Set

A famous fractal set is The Cantor Set is a famous fractal set consisting of points lying on a single line segment. It is formed by removing the middle third of each line segment with each iteration. Here the first several iterations of the Cantor Set are shown:



Initially, the line segment is defined on the interval $[0, 1]$. After the first iteration, the middle third of the segment $(\frac{1}{3}, \frac{2}{3})$ is removed leaving two line segments: $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. This process can be defined for the nth set by:

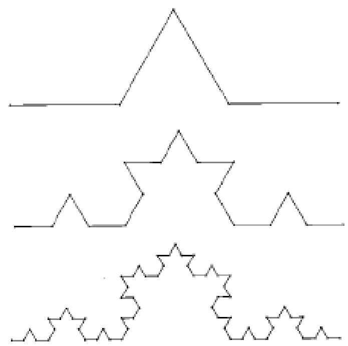
$$C = \bigcap_{n=1}^{\infty} C_n = \lim_{n \to \infty} C_n$$

$$C_n = \frac{C_{n-1}}{3} \cup (\tfrac{2}{3} + \tfrac{C_{n-1}}{3}) \text{ for } n \geq 1 \text{ and } C_0 = [0, 1].$$

As mentioned earlier, self-similar sets are more simple to find the dimension for. The dimension for a self-similar set can be calculated by $d = \frac{log(P)}{log(S)}$ where P represents the amount of self-similar copies present and S represents the scaling factor. For the Cantor Set, the dimension is equal to $\frac{log(2)}{log(3)}$.

2.5 The Koch Curve

The Koch Curve is another representation of a self-similar fractal set. Initially there is one line segment of length 1. After the first iteration, the middle third of the line segment is removed and two line segments equivalent to a third of the length of the original line segment are placed to form an equilateral triangle. This process if applied to each line segment for n iterations. A visual of the Koch Curve can be seen below.



K_n would represent the nth iteration of the Koch Curve and $K = \lim_{n\to\infty} K_n$. The perimeter of the curve is defined by, $\lim_{n\to\infty} K_n = \lim_{n\to\infty} 3s(\frac{4}{3})^n = \infty$, where s is the length of each line segment. Additionally, the area is equal to 0 since this is a curve. This implies there will be a dimension used between the values of 1 and 2. Since this is a self-similar set, the same method as the Cantor Set can be used to calculate the fractal dimension. Therefore, $d = \frac{log(4)}{log(3)} \approx 1.262$.

3 The Rudin-Shapiro Functions

3.1 The Rudin-Shapiro Sequence

The Rudin-Shapiro Sequence is introduced by Rudin[5] and Shapiro[6]. It is derived from the polynomial pairs, P_m, Q_m that are defined by:

$$P_0 = 1, Q_0 = 1$$

$$P_{m+1}(x) = P_m(x) + x^{2^m} Q_m(x)$$

$$Q_{m+1}(x) = P_m(x) - x^{2^m} Q_m(x)$$

Let (ε_k) be the kth coefficient of polynomial P_m, we have $\varepsilon_k = 1$ or -1. When $m \to \infty$, $(\varepsilon_0, \varepsilon_1, ...)$ is called the Rudin-Shapiro sequence. Rudin and Shapiro found that the exponential sum of the equation satisfies the following inequality:

$$max_\theta \mid \sum_{k=0}^{n-1} \varepsilon_k e^{ik\theta} \mid \leq (2+\sqrt{2})\sqrt{n}$$

Later, Saffari[7] was able to improve the coefficient of $(2+\sqrt{2})\sqrt{n}$ in the inequality to:

$$\max_\theta \mid \sum_{k=0}^{n-1} \varepsilon_k e^{ik\theta} \mid \leq \psi(\tfrac{log(n)}{log(2)})\sqrt{n} \leq (1+\sqrt{2})\sqrt{n}$$

by introducing a function ψ. The fractal dimensin of ψ was calculated to be $\frac{3}{2}$ in more general setting (see [2]). In this paper, we will show how ψ and two related functions Δ and phi were generated through a set E_k.

3.2 Forming the Set E_k

The set E_k (where k is an integer) represents the points that are being defined with the function. In this case, E, Δ, ϕ, and ψ are all defined on the interval $[1,2]$. The set E is defined by:

$$E_k = \{1 + \tfrac{d}{2^k}, d = 0, 1, .., 2^k\}, k \in \mathbb{Z}.$$

$$E = \bigcup_{k=0}^{\infty} E_k (\subset Q \cap [1,2]) = \lim_{x\to\infty} E_k$$

.

Initially E_0 is defined as $E_0 = \{1, 2\}$. The next iteration consists of the points from E_0 and the midpoint of the points in that set. So, $E_1 = \{1, \frac{3}{2}, 2\}$. This process is repeated for k iterations. To program this function, a program called Maple was used. The following code was used to create the set E_k.

```
with(plots) : with(LinearAlgebra) : with(DynamicSystems) :
N := 4 :
print( E[0] = [1, 2]) :
X := (i, k) → 1 + k·2^(-i) :
for i from 1 to N do
print(E[i] = [seq(X(i, k), k = 0 ..2^i)]) :
end do;
```

$$E_0 = [1, 2]$$

$$E_1 = \left[1, \frac{3}{2}, 2\right]$$

$$E_2 = \left[1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2\right]$$

$$E_3 = \left[1, \frac{9}{8}, \frac{5}{4}, \frac{11}{8}, \frac{3}{2}, \frac{13}{8}, \frac{7}{4}, \frac{15}{8}, 2\right]$$

$$E_4 = \left[1, \frac{17}{16}, \frac{9}{8}, \frac{19}{16}, \frac{5}{4}, \frac{21}{16}, \frac{11}{8}, \frac{23}{16}, \frac{3}{2}, \frac{25}{16}, \frac{13}{8}, \frac{27}{16}, \frac{7}{4}, \frac{29}{16}, \frac{15}{8}, \frac{31}{16}, 2\right]$$

Note that in Maple, the sets are defined with vector notation and represented by $[\cdot]$ instead of $\{\cdot\}$. Initially, the number of sets that are desired to be formed are denoted by $N := 4$. In this case, the sets through E_4 are shown. The number of sets the loop produced can be changed by increasing or decreasing the value of N. Afterwards, the initial function E_0 is defined, followed by the equation of the function E. From there, a loop is crated to show k number of sets in a sequence form.

3.3 Generating the Function Δ

Once the set E is defined, the function Δ can be formed. Δ can be defined by:

$$\Delta(1 + \tfrac{d}{2^k}) = 2^{-\frac{k}{2}} + \min(\Delta((d-1)2^{-k}), \Delta((d+1)2^{-k}))\ (1)$$

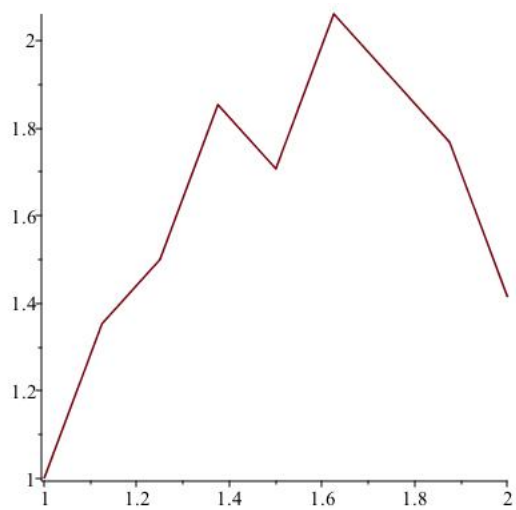
This function takes each value in a specific set E_k and corresponds them to a Δ value. For E_0, the delta values are initially defined as $\Delta(1) = 1$, $\Delta(2) = \sqrt{2}$. Then using the equation stated above, all the additional Δ values can be calculated. Using maple, the program to define the Δ values was then formed:

```
N := 3 :
D(1) := 1 :
D(2) := sqrt(2) :
print( E[0] = [1, 2]) :
X := (i, k) → 1 + k·2^(-i) :

for i from 1 to N do

print(E[i] = [seq(X(i, k), k = 0 ..2^i)]) :

for k from 1 to 2^i by 2 do
D(1) = 1;
D(2) = sqrt(2);

D(X(i, k)) := 2^(-i/2) + min(D(1 + (k - 1)2^(-i)), D(1 + (k + 1)2^(-i)));
end do;
for k from 0 to 2^i do

 Delta(X(i, k)) = D(X(i, k));

end do;
end do;
```

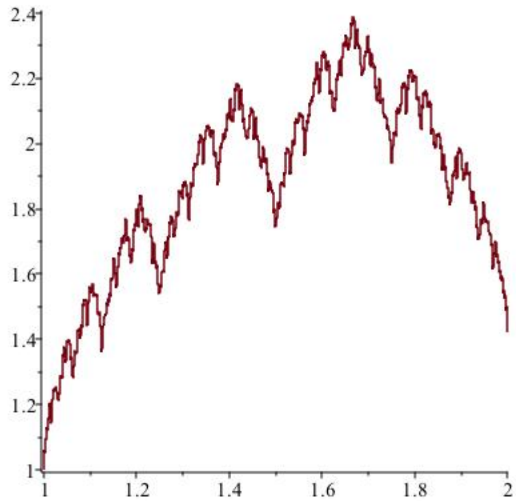
This program will generate the values for each set E_k. After the initial loop to acquire the values of E_k, another loop was implemented to get the Δ values. Inside the second k loop, the Δ values for E_0 were initially defined. Then the Δ equation was defined and printed. In this case, the Δ values up to the set E_3 were generated because $N = 3$. To graph the function, an additional program was used after the loop concluded:

```
E1 := k → 1 + (k − 1)·2^(-N) :
D1 := k → D(1 + (k − 1)·2^(-N)) :
Ns := 2^N + 1 :
E2 := Vector[row](Ns, E1) :
D2 := Vector[row](Ns, E2) :
P1 := Vector[row](N1, C) :
plot([seq([E2[i], D2[i]], i = 1 ..LinearAlgebra[Dimension](E2))]);
```

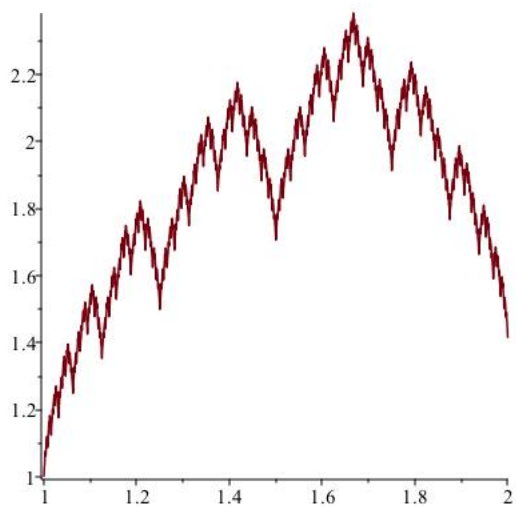
This program defines a vector $E2$ and a vector $D2$, from the equations of E and Δ. These two equations are denoted by $E1$ and $D1$. The function Δ is then graphed on E_N for a given N. It is important to note that the graphs for Δ are represented by a series of points. In this case, it becomes easier to observe the form the function takes if there are lines connecting the points. For $N = 3$ the following graph results in:



Its hard to form any conclusions about the Δ graph when the value of N is small. Once the value of N is increased to 10, the graph then becomes:



Similarly, when $N = 15$, the graph becomes even more defined:



As seen, the graphs for $N = 10$ and $N = 15$ look similar on the surface level. However, as the graphs are observed closer, the graph for $N = 15$ will have more defined points.

3.4 Generating the Functions ϕ and ψ

Now that the function Δ is found, then the function ϕ can be formed. ϕ is defined by the following function:

$$\phi(x) = \Delta(x)\sqrt{\tfrac{2}{x}}$$
$$\phi(2x) = \phi(x), \text{ where } x \in (0, \infty)$$

The function ϕ is a repeated function, where the function defined on the interval $[1, 2]$ is similar to the same function defined on $[2, 4]$, $[4, 8], ..., [2^{k-1}, 2^k]$ where $k \in \mathbb{Z}$. Once ϕ is found, ψ can be described by:

$$\psi(y) = \phi(2^y) \text{ for all } y \in \mathbb{R}$$

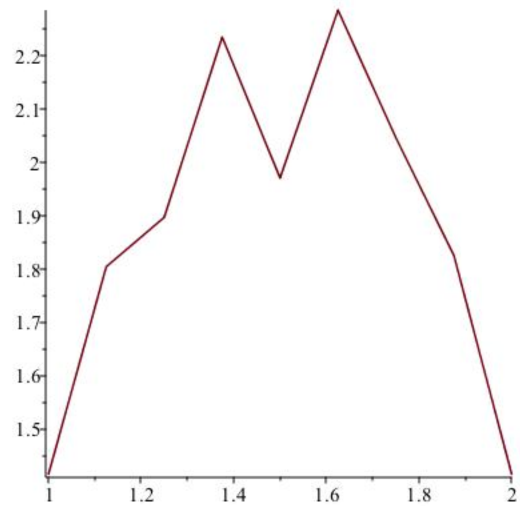
This shows that ψ is a periodic function with each period being equal to 1. For the graph of Δ and ϕ, the interval $[1, 2]$ was chosen as it was the simplest to observe the function's properties. The program of the function ϕ can be seen by:

```
for k from 0 to 2^N do
p(X(N, k)) := D(X(N, k))·sqrt(2/X(N, k)):
end do:
Phi[N] := [seq(p(X(N, k)), k=0..2^N)];
```

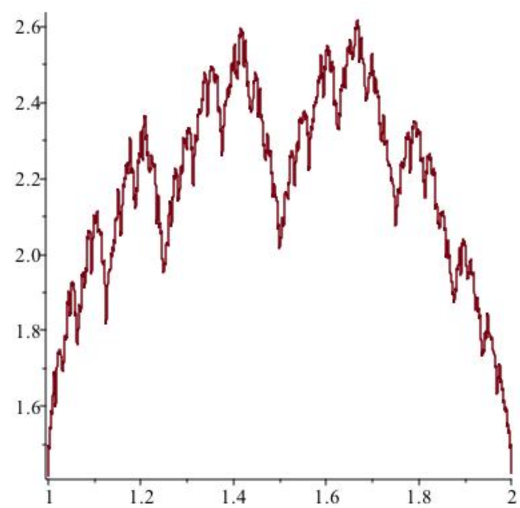
After the loop used for E and Δ an additional loop was created to form the function ϕ. Once the equation is defined inside the loop, the function ϕ is then printed out in sequence form. Once the values are calculated for ϕ a graph can then be formed similar to that of the Δ function.

```
E1 := k→1 + (k − 1)·2^(-N):
P1 := k→D(1 + (k − 1)·2^(-N))·sqrt(2/(1 + (k − 1)·2^(-N))):
Ns := 2^N + 1:
E2 := Vector[row](Ns, E1):
P2 := Vector[row](Ns, P1):
plot([seq([E2[i], P2[i]], i=1..LinearAlgebra[Dimension](E2))]);
```

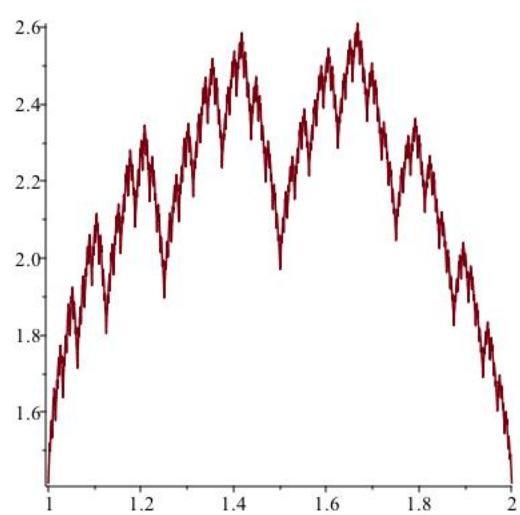
The function $\phi(x)$ id defined on E_k and graphed. The graph for $N = 3$ is seen below:



As N is increased, the graphs become more defined. Take $N = 10$:



Increasing the count of N to be 15 results in a more defined graph:



On the surface level, the graphs appear similar. If a more microscopic view is used, it will show the graph when $N = 15$ has a larger set of points than that of the $N = 10$ graph. Once again, the function is defined by a set of points. The line between the points helps visualize the form the function takes. As $N \to \infty$, the graph is represented by connecting lines because ϕ is dense in the interval $[1, 2]$. This means there are infinite points defined in the interval where the function is continuous everywhere and nowhere differentiable.

4 Rudin-Shapiro Fractal Dimension Calculation

Once the functions Δ, ϕ and ψ are defined, the fractal dimension can be calculated using the Box dimension method.

Theorem 1 in [2] states: Let $\Gamma(f, I) = \{x, f(x)), x \in I\}$ denote the graph of function f on interval I. Then,

$$\text{(i) } \dim_B \Gamma(\Delta, I) = \tfrac{3}{2}, \text{ (for all } I \subset (0, \infty)).$$
$$\text{(ii)} \dim_B \Gamma(\phi, I) = \tfrac{3}{2}, \text{ (for all } I \subset (0, \infty))$$
$$\text{(iii)} \dim_B \Gamma(\psi, I) = \tfrac{3}{2}, \text{ (for all } I \subset \mathbb{R})$$

The interval $(0, \infty)$ is used if I is extended. In this research the interval $[1, 2]$ was chosen.

Proof: Let $I_0 = [1, 2]$, then by Lemma 2 in [2] and Corollary 11.2 in [1], we can get $\overline{dim}(\Gamma(\Delta, I_0)) \leq \frac{3}{2}$ and $\overline{dim}_B(\Gamma(\phi, I_0)) \leq \frac{3}{2}$.

Then using Lemma 3 stated in [2]:

$$\text{For } 2^k \leq r < 2^{k+1} (i \in \mathbb{Z}, k = 0, 1, ...), \mid \Delta((r+1) \cdot 2^{-k}) - \Delta(r \cdot 2^{-k}) \mid = 2^{\frac{-k}{2}} \text{ or } (\sqrt{2} - 1) 2^{\frac{-k}{2}}.$$

Then,

$$\max_{x \in I_k} \Delta(x) - \min_{x \in I_k} \Delta(x) \geq \mid \Delta(r \cdot 2^{-k}) - \Delta((r+1) \cdot 2^{-k}) \mid \geq (\sqrt{2} - 1) 2^{-\frac{k}{2}}.$$

Additionally,

$$N_{\delta_k}(\Gamma(\Delta(I_0)) \geq (\sqrt{2} - 1) \cdot 2^{-\frac{k}{2}} \cdot 2^k \cdot 2^k = (\sqrt{2} - 1) \cdot 2^{\frac{3k}{2}},$$
$$\underline{dim}_B(\Gamma(\Delta, I_0)) = \underline{lim}_{k \to \infty} \frac{log N_{\delta_k}(\Gamma(\Delta(I))}{-log \delta_k} \geq \lim_{k \to \infty} \frac{log(\sqrt{2} - 1) 2^{\frac{3k}{2}}}{-log 2^{-k}} = \frac{3}{2}.$$

Also, $\underline{dim}_B(\Gamma(\phi, I_0) \geq \frac{3}{2}$. Similarly,

$$\underline{dim}_B(\Gamma(\phi, I_0)) \geq \frac{3}{2}.$$

Therefore, $dim_B\Gamma(\Delta, I_0) = \frac{3}{2}$ and $dim_B\Gamma(\phi, I_0) = \frac{3}{2}$. Lastly, by $\psi(y) = \phi(2^y)$,

$$\dim_B(\Gamma(\psi, I)) = \frac{3}{2}.$$

5 Conclusion

Calculating and graphing the functions Δ, ϕ and ψ in a set E is the culmination of this research at this point. The graphs were analyzed up to fifteen iterations. Anything past the fifteenth iteration would not be able to be processed through Maple. It would be interesting to see how the graphs could look on the hundredth or even thousandth iteration. This would allow the graph to be analyzed on a more finite scale; although it would require highly advanced graphing software. In addition, it would be useful to have the capability to view the graph on a more microscopic scale. As mentioned earlier, calculating the dimension provides the capability to calculate the measure of a fractal curve. If time permitted, another possible addition that could be made to this research would be to calculate the measure for this fractal function. These are just several of the possibilities on ways to potentially expand this research. Fractals are an intricate field of study in mathematics which provide countless pathways of research.

References

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