MODELING THE GAME PLINKO WITH RANDOM WALKS

DYLAN HOGAN

ABSTRACT. In this talk we model the popular The Price is Right game Plinko. We introduce a simple random walk and illustrate with a variation of Plinko. Further, we give a table of the probabilities of landing in specific slots that depends on starting position by using a random walk with reflecting barriers. We also investigate the effects of other Plinko boundaries and modify the length of the Plinko Board.

1. Introduction

Games of chance have always enticed people. The thought of winning fortune from a game is exhilarating. Television shows such as the Price is Right have revolutionized the way Americans can partake in these games. One of the most popular games on the show is called Plinko. Plinko is a game in which contestants take circular chips and slide them down a board that has rows of pegs in it, so that the chip eventually comes to rest in a slot at the bottom of the board, see Figure 1. These slots in the bottom of the Plinko board are all associated with different sums of money. Each contestant seems to have a different method of where to slide the chip to get the chip in the very best slot. In this paper we model Plinko with a random walk.

Simple random walks follow a binomial distribution normally, which we prove in Section 2. There are other examples of random walks that are of interest to us. By varying the game Plinko, we can look at other versions of these random walks. We can vary the game by changing the boundary cases; we start with a simple random walk with no barrier case, then we look at a case with reflecting barriers, and lastly absorbing barriers are investigated. We also vary the game by changing the length of the Plinko board.

Plinko is an example of a random walk with reflecting barriers. Referring to Figure 1, suppose a contestant drops one of their Plinko chips in to slot 5 on the Plinko board, and as it falls down it immediate goes

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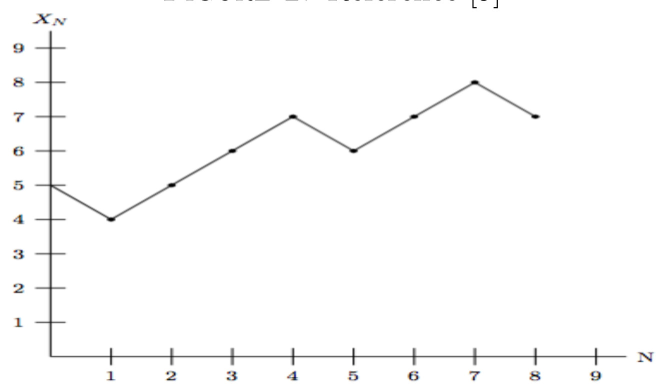
Figure 1. Reference: [8]

down to the left 8 times, which is possible because the probability of going left or right at any given time is 0.5. After immediately going left 8 times the chip would be at a wall, or reflecting barrier, of the Plinko board where the chip can not go left any more, thus the probability of going to the right would change from 0.5 to 1. We investigate the behavior further below.

2. Simple Random Walk

We can show random walks graphically in order to make the idea more clear. Figure 2, shows one possible path that a random walk could take. Observe that every point in the graph would be like a peg on the Plinko board, and every line represents the direction the chip would fall. In Plinko, the chips fall left or right down the board, but with a graphical representation each step is along the horizontal axis with the change in position done on the vertical axis.

FIGURE 2. Reference [9]



Following [5] we get the following definition:

Definition 2.1. *A simple random walk is a stochastic sequence $\{X_n\}$, with $X_0 = 0$, defined by*

$$X_n = \sum_{k=1}^{n} S_k,$$

where $\{S_k\}$ are independent and identically distributed random variables. Where $S_k = \pm 1$, with $P(S_k = 1) = p$ and $P(S_k = -1) = 1 - p = q$.

Let n_1 be the total number of times our simple random walk moves up, let n_2 be the number of times the walk moves down, and let N be the total number of steps in the random walk. Then:

$$n_1 + n_2 = N$$

and

$$n_1 - n_2 = K,$$

where K is the number of steps above or below our starting point. At this point we can make a couple other observations. Observe,

$$2n_1 = n + k \implies n_1 = \frac{N + K}{2}$$

where N and K have the same parity. In particular at each step X_N switch from even to odd, so if N and K are of different parity $P(X_N = K) = 0$.

Proposition 2.2. *Let p be the probability of going up, and q be the probability of going down. Let $p = P(S_K = 1)$ and $q = P(S_K = -1)$.*

$$P(X_N = K) = \binom{N}{n_1} p^{n_1} q^{n_2} = \binom{N}{\frac{N+K}{2}} p^{\frac{N+K}{2}} q^{\frac{N-K}{2}}$$

Proof. We will show this using induction. For our base case we have $P(X_1 = 1) = p$ and $P(X_1 = -1) = q$, following the definition.

$$P(X_1 = 1) = \binom{1}{1} p^1 q^0 = p$$

And,

$$P(X_1 = -1) = \binom{1}{0} p^0 q^1 = q$$

Now assume the condition holds for N. We substitute the values for n_1 and n_2, and the we need to show,

$$P(X_{N+1} = K) = \binom{N+1}{\frac{N+K+1}{2}} p^{\frac{N+K+1}{2}} q^{\frac{N-K-1}{2}}.$$

Observe that $P(X_{N+1} = K) =$

$$P(X_{N+1} = K | X_N = K-1) P(X_N = K-1) + P(X_{N+1} = K | X_N = K+1) P(X_N = K+1)$$

By the induction hypothesis, this gives:

$$p \binom{N}{\frac{N+K-1}{2}} p^{\frac{N+K-1}{2}} q^{\frac{N-K+1}{2}} + q \binom{N}{\frac{N+K+1}{2}} p^{\frac{N+K+1}{2}} q^{\frac{N-K-1}{2}}$$

.

Then by distributing the p and q,

$$= p^{\frac{N+K+1}{2}} q^{\frac{N-K-1}{2}} \left(\binom{N}{\frac{N+K-1}{2}} + \binom{N}{\frac{N+K+1}{2}} \right)$$

We almost have the desired result, we just need to show

$$\binom{N}{\frac{N+K-1}{2}} + \binom{N}{\frac{N+K+1}{2}} = \binom{N+1}{\frac{N+K+1}{2}}$$

Using the binomial expansion,

$$\binom{N}{\frac{N+K-1}{2}} + \binom{N}{\frac{N+K+1}{2}} = \frac{N!}{(\frac{N-K+1}{2})!(\frac{N+K-1}{2})!} + \frac{N!}{(\frac{N-K-1}{2})!(\frac{N+K+1}{2})!}$$

$$= \frac{N!}{(\frac{N-K+1}{2})(\frac{N-K-1}{2})!(\frac{N+K-1}{2})!} + \frac{N!}{(\frac{N-K-1}{2})!(\frac{N+K+1}{2})(\frac{N+K-1}{2})!}$$

We now get a common denominator and combine,

$$= \frac{N!(\frac{N+K+1}{2} + \frac{N-K+1}{2})}{(\frac{N-K+1}{2})(\frac{N-K-1}{2})!(\frac{N+K+1}{2})(\frac{N+K-1}{2})!}$$

$$= \frac{N!(N+1)}{(\frac{N-K+1}{2})!(\frac{N+K+1}{2})!}$$

$$= \frac{(N+1)!}{(\frac{N+1-K}{2})!(\frac{N+1+K}{2})!}$$

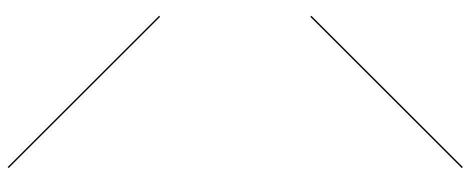
$$= \binom{N+1}{\frac{N+K+1}{2}}$$

Thus giving us the desired result. Combining the whole equation again then yields,

$$P(X_{N+1} = K) = \binom{N+1}{\frac{N+K+1}{2}} p^{\frac{N+k+1}{2}} q^{\frac{N-K-1}{2}}.$$

□

A true simple random walk cannot be represented in a matrix, because a simple random walk is infinite. Since Plinko is a finite example, we can represent it with a matrix. First consider the case with no boundary conditions, where Plinko is a simple random walk. The physical Plinko boundaries would then look like:



This boundary behavior is such that the bounds never come in to effect. As stated earlier, N and K must have the same parity, hence on a graph, the middle of the Plinko board would be on the horizontal axis when $x = 10$. Since Plinko has nine possible starting slots and graphically they would have to be even, a matrix representation would have to be very large, in fact, a 37×37 matrix. This is because our chip would not be limited to nine ending slots, but actually anywhere $-8 \leq X_{13} \leq 28$. Consider the matrix, $P = (p_{ij})$, where $p_{ij} = P(X_n = i|X_{n-1} = j)$ and p_{ij} is the transition probability. We can show this matrix as:

$$P = \begin{bmatrix} 0 & q & 0 & 0 & \dots & 0 & 0 & 0 \\ p & 0 & q & 0 & \dots & 0 & 0 & 0 \\ 0 & p & 0 & q & \dots & 0 & 0 & 0 \\ . & . & . & . & \dots & . & . & . \\ . & . & . & . & \dots & . & . & . \\ . & . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & 0 & \dots & p & 0 & q \\ 0 & 0 & 0 & 0 & \dots & 0 & p & 0 \end{bmatrix}.$$

Making a 37×1 initial vector,

$$U = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ 1 \\ 0 \\ . \\ . \\ 0 \\ 0 \end{bmatrix}$$

we can find the probability of any possible ending slot by doing the computation of $P^{12}(U)$ This result is the same as that predicted by Proposition 2.2.

3. Reflecting Barriers

A random walk with reflecting barriers is easier to see in a matrix representation. Continuing with the trend, let p and q be as defined above. The matrix representation would then be, square matrix $R = (p_{ij})$, where $p_{ij} = P(X_n = j | X_{n-1} = i)$,

$$R = \begin{bmatrix} 0 & q & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & q & 0 & \dots & 0 & 0 & 0 \\ 0 & p & 0 & q & \dots & 0 & 0 & 0 \\ . & . & . & . & \dots & . & . & . \\ . & . & . & . & \dots & . & . & . \\ . & . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & 0 & \dots & p & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & p & 0 \end{bmatrix}.$$

A random walk with reflecting barriers can also explain our final slot Plinko probabilities. By taking our above matrix R and raising it to the $12th$ power and multiplying by a unit vector we can get Plinko's probabilities. Let U be a unit vector of size n_R,

$$U = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

where n_R = the number of rows in matrix R. This would also imply that n_R = 2x the number of slots that the Plinko chip could land in. Notice that every entry in U is 0 except for one entry containing 1. This represents our starting location of the Plinko chip. $R(U)$ = the position after one step, $R^2(U)$ = the position after two steps and $R^{12}(U)$ = the position of the Plinko chip at the bottom. Let $p = q = \frac{1}{2}$. This yields the following table which is also given in [5]. This table is identical to the one in [5], but the values are computed differently.:

Ending Position	$100	$500	$1,000	$0	$10,000	$0	$1,000	$500	$100
Slot 1	0.2256	0.3867	0.2417	0.1074	0.0322	0.0059	0.0005	0	0
Slot 2	0.1934	0.3464	0.2471	0.137	0.0566	0.0164	0.0029	0.0002	0
Slot 3	0.1208	0.2471	0.2417	0.1963	0.1211	0.0537	0.0161	0.0029	0.0002
Slot 4	0.0537	0.137	0.1963	0.2258	0.1934	0.1208	0.0537	0.00164	0.0029
Slot 5	0.0161	0.0566	0.1211	0.1934	0.2256	0.1934	0.1211	0.0566	0.0161
Slot 6	0.0029	0.0164	0.0537	0.1208	0.1934	0.2258	0.1963	0.137	0.0537
Slot 7	0.0002	0.0029	0.0161	0.0537	0.1211	0.1963	0.2417	0.2471	0.1208
Slot 8	0	0.0002	0.0029	0.0164	0.0566	0.137	0.2471	0.3464	0.1934
Slot 9	0	0	0.0005	0.0059	0.0322	0.1074	0.2417	0.3867	0.2256

This matrix could be nearly infinite in dimension, this creating a good guess at the distribution. As the matrix gets taller, the distribution will then begin to become more uniform as the bounds play a larger and larger roll in the final stopping point of the random walk.

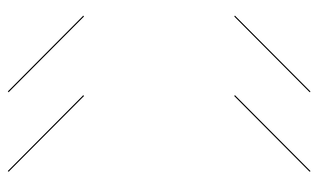
We now let N=1000, in order to observe the case of N approaching infinity.

Ending Position	$100	$500	$1,000	$0	$10,000	$0	$1000	$500	$100
Slot 1	0.0625	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.0625
Slot 2	0.0625	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.0625
Slot 3	0.0625	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.0625
Slot 4	0.0625	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.0625
Slot 5	0.0625	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.0625
Slot 6	0.0625	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.0625
Slot 7	0.0625	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.0625
Slot 8	0.0625	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.0625
Slot 9	0.0625	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.0625

From the table above, we can see that, experimentally, the distribution becomes uniform. This result makes sense, because the boundaries come in to play so much that it does not matter what slot the Plinko chip is dropped in.

4. Absorbing Barriers

The boundary cases are not always continuous. Consider the case where the boundaries look like:



This would be the case of the absorbing barriers, where if the boundary came in to play, it would immediately leave the board and fall into one of the boundary slots at the bottom. This case can also be represented using a matrix. The matrix then becomes,

$$S = \begin{bmatrix} 1 & q & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & q & 0 & \dots & 0 & 0 & 0 \\ 0 & p & 0 & q & \dots & 0 & 0 & 0 \\ . & . & . & . & \cdots & . & . & . \\ . & . & . & . & \cdots & . & . & . \\ . & . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & 0 & \dots & p & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & p & 1 \end{bmatrix}.$$

A random walk with absorbing barriers is different from a random walk with reflecting barriers in that the boundaries represent a stopping point. If our example of the Plinko board had absorbing barriers instead of reflecting barriers, the chip would reach the boundary and immediately fall to the slot directly below, instead of bouncing back in to the board. Using the same method as with reflecting barriers with $p = q = \frac{1}{2}$, we can see what the Plinko probabilities would look like:

Ending Position	$100	$500	$1,000	$0	$10,000	$0	$1000	$500	$100
Slot 1	1	0	0	0	0	0	0	0	0
Slot 2	0.5811	0.1047	0.1396	0.1047	0.0508	0.0159	0.0029	0.0002	0
Slot 3	0.2668	0.1396	0.2095	0.1904	0.1206	0.0537	0.0161	0.0029	0.0002
Slot 4	0.0923	0.1047	0.1904	0.2253	0.1934	0.1208	0.0537	0.0159	0.0034
Slot 5	0.0225	0.0508	0.1206	0.1934	0.2256	0.1934	0.1206	0.0508	0.0225
Slot 6	0.0034	0.0159	0.0537	0.1208	0.1934	0.2253	0.1904	0.1047	0.0923
Slot 7	0.0002	0.00293	0.0161	0.0537	0.1206	0.1904	0.2095	0.1396	0.2668
Slot 8	0	0.0002	0.0029	0.0159	0.0508	0.1047	0.1396	0.1047	0.5811
Slot 9	0	0	0	0	0	0	0	0	1

This is interesting because the middle slot probabilities are fairly similar to those of the reflecting barriers, but the closer to the barrier the more the absorption takes affect.

Next we let N = 100 to see what would begin to happen as N approached infinity. When N=100 we get the following table of values:

Ending Position	$100	$500	$1,000	$0	$10,000	$0	$1000	$500	$100
Slot 1	1	0	0	0	0	0	0	0	0
Slot 2	0.8403	0.0053	0.0098	0.0127	0.0137	0.0127	0.0097	0.0052	0.0905
Slot 3	0.686	0.0098	0.0181	0.0235	0.0254	0.0234	0.0179	0.0097	0.1863
Slot 4	0.5415	0.0127	0.0235	0.0307	0.0332	0.0306	0.0234	0.0127	0.2917
Slot 5	0.4097	0.0137	0.0254	0.0332	0.0359	0.0332	0.0254	0.0137	0.4097
Slot 6	0.2917	0.0127	0.0234	0.0306	0.0332	0.0307	0.0235	0.0127	0.5415
Slot 7	0.1863	0.0097	0.0179	0.0234	0.0254	0.0235	0.0181	0.0098	0.686
Slot 8	0.0905	0.0052	0.0097	0.0127	0.0137	0.0127	0.0098	0.0053	0.8403
Slot 9	0	0	0	0	0	0	0	0	1

With this table, it is evident that as N gets larger, the absorbing barriers are much more prevalent. To further show the point let N=1000:

Ending Position	$100	$500	$1,000	$0	$10,000	$0	$1000	$500	$100
Slot 1	1	0	0	0	0	0	0	0	0
Slot 2	0.875	0	0	0	0	0	0	0	0.125
Slot 3	0.75	0	0	0	0	0	0	0	0.25
Slot 4	0.625	0	0	0	0	0	0	0	0.375
Slot 5	0.5	0	0	0	0	0	0	0	0.5
Slot 6	0.375	0	0	0	0	0	0	0	0.625
Slot 7	0.25	0	0	0	0	0	0	0	0.75
Slot 8	0.125	0	0	0	0	0	0	0	0.875
Slot 9	0	0	0	0	0	0	0	0	1

At this point, there is virtually no chance to land anywhere but the barrier case.

5. Conclusion and Future Directions

In the future we would like to take our two dimensional Plinko board and make it in to a three dimensional cube. In three dimensions, instead of the chip going in two directions, left and right, the chip would be able to go down in four directions, left, right, forward, and backward. Additional research should look at a 2-dimensional random walk, and the probabilities that would yield in different boundary cases.

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Dylan Hogan, Mathematics Department, Georgia College, Milledgeville, GA 31061, U.S.A.

E-mail address: james.hogan@bobcats.gcsu.edu