Sophie Germain, The Princess of Mathematics and Fermat's Last Theorem

Hanna Kagele

Abstract

Sophie Germain (1776-1831) is the first woman known who managed to make great strides in mathematics, especially in number theory, despite her lack of any formal training or instruction. She is best known for one particular theorem that aimed at proving the first case of Fermats Last Theorem. Recent research on some of Germain‘s unpublished manuscripts and letters reveals that this particular theorem was only one minor result in her grand plan to prove Fermat‘s Last Theorem. This paper focuses on presenting some of Sophie Germains work that has likely lain unread for nearly 200 years.

1. Introduction

Sophie Germain has been known for years as the woman who proved Sophie Germain's Theorem and as the first woman who received an award in mathematics. These accomplishments are certainly impressive on their own, especially being raised in a time period where women were discouraged from being educated. However, it has recently been discovered that her work in number theory was far greater than an isolated theorem. In the last 20 years, her letters to mathematicians such as Gauss and Legendre have been analyzed by various researchers and her extensive work on the famous Fermat's Last Theorem has been uncovered. Germain was the first mathematician to ever formulate a cohesive plan for proving Fermat's Last Theorem. She worked tirelessly for years at successfully proving the theorem using her method involving modular arithmetic. In pursuit of this goal, she also proved many other results, including the theorem she is famous for. However, she never chose to publish any of her work. Until very recently, anything known about her work in number theory was credited to her by other famous mathematicians such as Legendre. To make her accomplishments even more incredible, Germain did all of this with no formal education whatsoever. Her passion for mathematics, specifically number theory, even when tangible results seemed difficult to attain, is truly inspiring. In this paper, we will first discuss some of Sophie Germain's background and mathematical milestones. Then we will discuss Fermat's Last Theorem and why it is significant that Sophie Germain devoted her attention to it. Then some preliminaries that are necessary in Germain's work are presented. Finally, we will discuss Germain's grand plan to prove Fermat's Last Theorem, which includes both known work and recently discovered work.

2. Sophie Germain

Germain has a fascinating story that tells of her self-motivating, passionate and determined character even with the obstacle of being a female in the 18th and 19th centuries. Germain was born in Paris in 1776 and spent her later childhood years in the midst of the French Revolution, living in the center of a war zone. The house she grew up in was only feet away from violent protests and marches that happened during the Revolution. Starting at age 13, Germain was confined to her home because of the dangers of the war, which gave her an ample amount of time to explore and discover what was in her father's library. The first mathematical text that Germain read was *Histoire des mathematiques*, in which she read about the story of Archimedes who was killed because he was so distracted and fascinated with mathematics that no one could stop him from practicing it [5]. Germain was inspired by this story and her own passion for mathematics was sparked. She began reading and studying more books to learn the subject that she was now encapsulated with. Despite her desire to educate herself, Germain's parents did not want her to be educated because they found it inappropriate for a woman of her class to do so. When her parents protested her studies, she began sneaking into the family library at night and reading textbooks over candlelight. Her parents went as far as to give her no heat or clothing at night so that she would stay in bed, but this still didn't stop young Germain from spending all night researching. Her parents then realized that they couldn't keep her from pursuing mathematics and they ceased their protests. Unique from most famous mathematicians up to this point, she had no private tutor or outside training; she was learning concepts as difficult as differential calculus on her own [7].

Germain continued studying independently until she was 18, when she connected to a professor at a new and prestigious French University, Ecole Polytechnique. During this time period, females were not allowed to attend lectures at the school, but Germain was determined to further her mathematical ability. She found a way to obtain lecture notes from the many well-known mathematicians who taught there. She also submitted some of her own personal work to a professor named Lagrange, who had assigned homework that was recorded in his lecture notes. Germain submitted this homework under the male name Le Blanc, who had previously attended the school but passed away. Her submission caught the attention of Lagrange, who insisted on meeting the author. Lagrange then learned that Le Blanc was actually Sophie Germain, a female, and was greatly impressed by her. He respected her ability and talent and brought her into his circle of scientists and mathematicians, an opportunity she had never gotten before. This motivated her even more in her mathematical research. Her relationship with Lagrange did not mark the end of Germain's correspondence with mathematicians under a hidden identity. In 1804, when she was 28 years old, Germain gained an interest in number theory and specifically Fermat's Last Theorem. She read both Adrien-Marie Legendre's and Carl Gauss's work regarding number theory and reached out to both of them. She began correspondence with Gauss about both his findings in number theory and the proof of Fermat's Last Theorem, again choosing to use Le Blanc's name instead of revealing her identity as a woman. Gauss was impressed with the work of Le Blanc, writing about "him" to his friend Wilhelm Olbers: "I am amazed that M. Le Blanc has completely mastered my *Disquisitiones arithmeticae*, and has sent me very respectable communications about them" [5]. Gauss found out Germain's true identity three years into their correspondence and he responded to this surprise in the same way that Lagrange did-with adoration and great respect for her. He wrote about her again, saying "But when a woman, because of her sex, our customs and prejudices, encounters infinitely many more obstacles than men, in familiarizing herself with their knotty problems, yet overcomes these fetters and penetrates that which is most hidden, she doubtless has the most noble courage, extraordinary talent, and superior genius" [5]. In 1808, four years into their correspondence, Gauss took a new job as an astronomy professor and his wife passed away, so he stopped responding to Germain's letters. Germain then turned her attention to other fields of mathematics for a number of years, until a prize was offered for a correct proof of Fermat's Last Theorem in 1815. This motivated her to begin working on the plan she had begun to develop years earlier. In need of a more experienced number theorist, she then began communication with Legendre as she continued working on the proof. In 1819 she wrote to Gauss one more time in hopes that he would respond. She shared with him her main idea to prove the Fermat's Last Theorem that showed how well she grasped some complex ideas in Gauss' work. Even though Germain made strides in proving Fermat's Last Theorem that eventually led other mathematicians to draw more conclusions with regard to the proof, she never chose to publish any of her efforts. Her work has long been known as a footnote of Legendre's publication about Fermat's Last Theorem, until a few mathematicians began to dig deeper into Germain's research in recent years. From these people examining Germain's letters more carefully, we know that Germain had a very involved plan to prove Fermat's Last Theorem, proved Case 1 of Fermat's Last Theorem for all odd primes less that or equal to 100, showed that any counterexamples to the theorem would have to be at least 30 digits long, and her theorem was later used to prove Fermat's Last Theorem for odd primes less than 1700 [7].

Because of Germain's talent and love for mathematics, Gauss requested that she be given an honorary degree from the University of Gottingen. Unfortunately, she died at age 55 to breast cancer before she could receive it. Even though the ultimate goal of proving Fermat's Last Theorem was not fully accomplished by Germain, her work on this proof influenced other mathematicians as well as the study of number theory. Germain made strides in other fields besides mathematics as well. She won an award from the Paris Academy of Sciences for her work on the theory of elasticity in physics and she published two philosophical papers. Her work in number theory, specifically on Fermat's Last Theorem, is not as widely known as some of the other work she completed, but this makes her efforts no less respectable or extraordinary [7].

3. Fermat's Last Theorem

In 1637, 139 years before Sophie Germain was born, Pierre de Fermat was a 28 year old lawyer who found pleasure in mathematics as a hobby. He was reading through Diophantis' book, *Arithmetica*, when he scribbled a note in the margin of the book that would impact mathematics forever. The note stated, in Fermat's language, that *"it is impossible to split any cube into two cubes, or a biquadrate into two biquadrates, or in general any power higher than the second, can be split into two powers of the same degree; I found a marvelous proof that this margin is too small to contain."* [8]. In modern language, Fermat's Last Theorem is stated as "no three positive integers x, y, z will satisfy the equation

$$x^n + y^n = z^n$$

for integers $n > 2$". This equation should look familiar to anyone that has been taught foundational mathematics since the $n = 2$ case is famously the Pythagorean Theorem, which has infinitely many integer solutions. The Pythagorean Theorem was known long before Fermat conjectured his statement about exponents greater than 2.

Fermat claimed to have found a proof that was too large to fit in the margin of *Arithmetica*. A proof by Fermat (other than for exponent four) was never found, so it is still not known whether he actually did have a successful proof or not, but most mathematicians agree that if there was a proof it was probably not complete. For 358 years a number of mathematicians, including Sophie Germain, attempted to prove this theorem that may seem simple at first glance. By the 1800s this theorem was known as mathematics' most difficult theorem to prove. It wasn't until 1954 that a professor named Andrew Wiles discovered a proof of Fermat's Last Theorem dealing with algebraic geometry [4]. If all of the work done on Fermat's Last Theorem by mathematicians through history were compiled into one stack of papers, this stack would be approximately three meters tall [8].

4. Preliminaries

There are many results that Germain uses in her work that she does not distinctly prove. These are proved as follows.

Lemma 1 If the product of two relatively prime positive integers is an n^{th} power, then each of the integers is an n^{th} power.

Proof. Let x and y be relatively prime positive integers and let $xy = a^n$ where $a, n \in \mathbb{Z}^+$. Note that when an n^{th} power is broken down into its prime factorization, all exponents must be divisible by n. That is,

$$a^n = (p_1^{\alpha_1} ... p_k^{\alpha_k})^n = p_1^{n\alpha_1} ... p_k^{n\alpha_k},$$

where p_i is prime and $\alpha_i \in \mathbb{Z}^+$ for $i = 1, ..., k$. Thus, if in the prime factorization of a number, all exponents are divisible by n, the number itself is an n^{th} power. The primes p_i in $xy = a^n = p_1^{n\alpha_1} ... p_k^{n\alpha_k}$ are the same primes as in the prime factorization of x and y. So each p_i must be in the prime factorization of either x or y, and if a prime is in the prime factorization of x or y, it must be one of p_i. Since x and y are relatively prime, they have no common primes in their prime factorizations. That is, if p_i is a factor of x then $p_i^{na_i}$ is a factor of x. Similarly, if p_i is a factor of y then $p_i^{na_i}$ is a factor of y. Therefore, every exponent in the prime factorization of both x and y must be divisible by n, so x and y are both n^{th} powers. □

Lemma 2 It is sufficient to prove FLT for odd prime exponents.

Proof. Consider the equation in FLT: $x^p + y^p = z^p$ where $p > 2$. Note that p can be broken down into its prime factorization. That is, $p = q_1 q_2 ... q_n$ where each q is prime and not necessarily unique. Then we have that

$$\begin{aligned} z^p &= x^p + y^p \\ z^{q_1 \cdots q_n} &= x^{q_1 \cdots q_n} + y^{q_1 \cdots q_n} \\ (z^{q_1 \cdots q_{k-1} q_{k+1} \cdots q_n})^{q_k} &= (x^{q_1 \cdots q_{k-1} q_{k+1} \cdots q_n})^{q_k} + (y^{q_1 \cdots q_{k-1} q_{k+1} \cdots q_n})^{q_k} \\ (z')^{q_k} &= (x')^{q_k} + (y')^{q_k} \end{aligned}$$

where $z' = z^{q_1 \cdots q_{k-1} q_{k+1} \cdots q_n}, x' = x^{q_1 \cdots q_{k-1} q_{k+1} \cdots q_n}, y' = y^{q_1 \cdots q_{k-1} q_{k+1} \cdots q_n}$ and q_k is prime. This is now a Fermat Equation with a prime exponent. Each prime exponent in the prime factorization can be factored out and written as a new version of the Fermat equation.
Every integer $p > 2$ is divisible by 4 or by an odd prime number (or both). Therefore, FLT cab be

proved for p if it can be proved for $q = 4$ or for all primes q. Since FLT was already proved for $q = 3$ and $q = 4$, it is sufficient to prove FLT for odd prime exponents.

□

Lemma 3 It is sufficient to prove FLT for x, y, z that are coprime.

Proof. Assume that x, y, z are not coprime. Thus, there exists an integer q that divides at least two of x, y, z.

Assume that q divides two of x, y or z. Without loss of generality, let $q|x$ and $q|z$. Then there exist $k_1, k_2 \in \mathbb{Z}$ such that $x = qk_1$ and $z = qk_2$. Then we have that

$$\begin{aligned} z^p &= x^p + y^p \\ (qk_2)^p &= (qk_1)^p + y^p \\ q^p k_2^p &= q^p k_1^p + y^p \\ y^p &= q^p k_2^p - q^p k_1^p \\ y^p &= q(q^{p-1} k_2^p - q^{p-1} k_1^p) \end{aligned}$$

Therefore, $q|y^p$. It follows that $q|y$. Therefore, if q divides two of x, y or z it must also divide the third.

Similarly, when q divides all three of x, y, z, then we have that

$$\begin{aligned} z^p &= x^p + y^p \\ (qk_3)^p &= (qk_1)^p + (qk_2)^p \\ q^p k_3^p &= q^p k_1^p + q^p k_2^p \\ k_3^p &= k_1^p + k_2^p, \end{aligned}$$

where k_1, k_2 and k_3 have no common divisors and are therefore coprime. Therefore, the Fermat equation can always be written in terms of variables that are coprime. Thus, it is sufficient to prove FLT for x, y, z that are coprime. □

Lemma 4 If the Fermat Equation for exponent p has a solution, and if θ is a prime number with no nonzero consecutive *pth* power residues modulo θ, then θ necessarily divides one of x, y or z [6].

Proof. We will prove this by contradiction. Assume that θ does not divide any of x, y or z.

Then we know that each of x, y and z have an inverse modulo θ. Without loss of generality, let a be the multiplicative inverse of x modulo θ. Then we have that

$$(az)^p \equiv (ax)^p + (ay)^p (mod\, \theta)$$

$$(az)^p \equiv 1 + (ay)^p \ (mod\, \theta)$$

Thus, the residues of $(az)^p$ and $(ay)^p$ are consecutive, a contradiction to the assumption. Therefore, θ divides one of x, y or z. □

5. Sophie Germain's plan to prove Fermat Last Theorem

Germain's Published Work

Only a tiny fraction of Sophie Germain's work has been known and credited to her up until the early 2000s when mathematicians including Andrea Centina, David Pengelley and Reinhard Laubenbacher began to research unpublished manuscripts written by Germain held in Italy and France [3],[6]. She is most famously known for a result that is now referred to as Sophie Germain's Theorem, which Legendre wrote about in 1823 and credited to Germain in the footnote of his own publication. This theorem is stated currently as follows.

Germain's Theorem For an odd prime exponent p, if there exists an auxiliary prime θ such that there are no two nonzero consecutive p-th powers modulo θ, nor is p itself a p-th power modulo θ, then in any solution to the Fermat equation $z^p = x^p + y^p$, one of x, y, or z must be divisible by p.

Germain proved the previous result in a letter that she wrote to Legendre about her work on Fermat's Last Theorem. She included two proofs, one in which the conclusion is that x, y or z must be divisible by p, as stated above, and one in which the conclusion is that x, y or z is divisible by p^2. The proof for the latter conclusion, which is known as the "stronger" version of her theorem, was incomplete in the letter. This is why Legendre credited her with the "weaker" version (the p case) in his own work. Later work by Germain shows that she correctly proved the stronger case after catching her own error, but most mathematicians still state Sophie Germain's Theorem as the weaker case since that is what Legendre published. The following proof is for the well-known case with the conclusion that x, y or z divides p.

Proof. We will prove by contradiction. Assume that p does not divide any of x, y or z. By Lemma 3, it is sufficient to prove this for x, y, z that are coprime.

Since p is odd, we can factor the equation $z^p = x^p + y^p$ as follows:

$$z^p = x^p + y^p = (x + y)(x^{p-1} - x^{p-2}y + ... + y^{p-1})$$

Now it is necessary to show that $(x + y)$ and $(x^{p-1} - x^{p-2}y + ... + y^{p-1})$ are coprime.

We will prove this by contradiction. That is, we will assume there exists a prime q that divides both $(x + y)$ and $(x^{p-1} - x^{p-2}y + ... + y^{p-1})$. Then $y = -x + jq$ where $j \in \mathbb{Z}$, and $(x^{p-1} - x^{p-2}y + ... + y^{p-1}) = px^{p-1} + dq$ where $d \in \mathbb{Z}$. We assumed that $(x^{p-1} - x^{p-2}y + ... + y^{p-1})$ is divisible by q, so it must be that px^{p-1} is divisible by q. From this we deduce that either p or x is divisible by q.

<u>Case 1:</u> If p is divisible by q, then p and q must be equal since they are both prime. Then, since $q|(x + y)$ by assumption, $p|(x + y)$, and also $p|z$. This contradicts the assumption that p is prime to x, y, z.

Case 2: Assume x is divisible by q. Then q is a factor of both x and $(x+y)$, contradicting the fact that x, y, z are coprime.

Thus, $(x+y)$ and $(x^{p-1} - x^{p-2}y + ... + y^{p-1})$ are coprime.

Therefore, by Lemma 1, $z^p = x^p + y^p = (x+y)(x^{p-1} - x^{p-2}y + ... + y^{p-1}) = (l^p)(r^p) = (lr)^p$. That is, $z = (lr)$. Following this same process using $x^p = z^p - y^p$ and $y^p = z^p - x^p$, we have that

$$x^p = z^p - y^p = (z-y)(z^{p-1} + z^{p-2}y + ... + y^{p-1}) = (h^p)(n^p) = (hn)^p,$$

$$y^p = z^p - x^p = (z-x)(z^{p-1} + z^{p-2}x + ... + x^{p-1}) = (v^p)(m^p) = (vm)^p.$$

where $l, r, h, n, v, m \in \mathbb{Z}$. So, $x = hn$ and $y = vm$.

From Lemma 4, we know that one of x, y, z is a multiple of θ. We will assume that it is z, without loss of generality.

Note that $(l^p) + (h^p) + (v^p) = (x+y) + (z-y) + (z-x) \equiv -2z \equiv 0 \bmod \theta$.

By Lemma 4, we have that one of l, h, v are a multiple of θ. If either h or v were a multiple of θ, then either y or x would have the factor θ, a contradiction to x, y, z being relatively prime since z has factor θ by assumption.
Then we will consider if l is multiple of θ. That is, $x + y = l^p \equiv 0 \bmod \theta$. This tells us that $y \equiv -x \bmod \theta$. Then $r^p = (x^{p-1} - x^{p-2}y + ... + y^{p-1}) \equiv px^{p-1} \equiv p(-y)^{p-1} \equiv p(-v^p)^{p-1} \equiv pv^{p(p-1)} \bmod \theta$. Dividing by $v^{p(p-1)}$, we get

$$p \equiv \left(\frac{r}{v^{p-1}}\right)^p.$$

This contradicts the assumption that there are no p^{th} power residues modulo θ. Thus, p must divide one of x, y or z.

□

Germain used this theorem for primes up to 100 using tables of calculations, as did Legendre after she shared it with him. An example of this calculation for $p = 3$ is as follows.

To satisfy the conditions of the theorem, we must find an auxiliary prime θ such that 3 is not a $3rd$ power modulo θ and there are no two consecutive $3rd$ power residues modulo θ. We will consider $\theta = 13$. The following table shows the cubic residues modulo 13 for each residue from 1 to 12.

For $\theta = 13$:

residue	1	2	3	4	5	6	7	8	9	10	11	12
cube	1	8	27	6	125	216	343	512	729	1000	1331	1728
cubic residue	1	8	1	12	8	8	5	5	1	12	5	12

Considering each cubic residue, we observe that 3 is not a cubic residue and no two cubic residues have a difference of one. Thus, the conditions for Sophie Germain's Theorem are satisfied and we know that any solution to the equation $x^3 + y^3 = z^3$ must be divisible by $p = 3$ [6].

Germain's Unpublished Work

Germain's First Correspondence with Gauss

Germain wrote her first letter to Gauss in the year 1804. In this letter, she claimed to have proven that the Fermat equation has no integer solutions for exponent $n = p - 1$ where p is a prime of the form $8k + 7$. Her proof in this letter was extensive and she must have worked on it for a long period of time before sending it to Gauss. Unfortunately, her proof was incomplete. Although, she did successfully prove other claims regarding these prime exponents that were necessary to her proof [3]. These include:

> *Let p be a prime, $p > 3$. If x, y and z are relatively prime integers such that $x^{p-1} + y^{p-1} = z^{p-1}$, then p divides the even one of x and y, moreover z is odd and not divisible by p.*

> *Let $p > 3$ be a prime. If x, y and z are relatively prime integers, and $x^{p-1} + y^{p-1} = z^{p-1}$ is satisfied, then x is a quadratic residue (mod p).*

Both of these results were also proved later by mathematicians Gandhi and Raina in 1966 and 1969, respectively [3].

Gauss wrote back to Germain six months later, not providing any comments on the proof contained in the letter. They continued correspondence until 1809, when Gauss began to deal with some personal issues and became more occupied with his teaching astronomy. At this point, Germain began working on other fields including vibrating surfaces. It wasn't until 1819 that Germain wrote to Gauss again about Fermat's Last Theorem.

Germain's Grand Plan

Sophie Germain was the first mathematician to ever formulate a plan to prove Fermat's Last Theorem in its entirety. She revealed this plan to Gauss in a twenty page letter, but this plan was not widely known until the recent work of David Pengeley and Reinhard Laubenbacher, who analyzed this letter [5]. The number theory results that she proved were always in pursuit of this goal, even the famous theorem named after her. She sought to prove that the Fermat Equation could not be solved for any odd prime exponent (the only necessary case not yet proven). Germain successfully proved that if an auxiliary prime of the form $2Np + 1$ (where p is prime) exists such that there are no two p^{th} powers modulo θ that are consecutive and and p itself is not a p^{th} power modulo θ, then this auxiliary prime will have to divide one of x, y, z where (x, y, z) is a solution to the Fermat equation. She found such an auxiliary prime for all exponents up to one hundred. She then sought to prove that there are an infinite number of these auxiliary primes for each exponent. If this were true, then an infinite number of primes would divide an integer, which is impossible. Germain had an incredible understanding of this equation and of modular arithmetic in order to formulate a plan of this complexity [6].

Once Germain had proved that if a certain auxiliary prime exists then p must divide x, y or z, she focused her attention on generating these auxiliary primes and searching for a proof that infinitely many of these auxiliary primes exist. She studied the non-consecutive condition and the relationship between this condition and any auxiliary prime. Germain was hopeful that her plan to produce infinitely many auxiliary primes would produce the first successful proof for Fermat's Last Theorem, but she began to realize that she had not even found one exponent for which infinitely many auxiliary primes exist, let alone all exponents. She found as many auxiliary primes as she could using only her calculations, realizing that any solution to the Fermat Equation would have to be very large in size. For example, she wrote that for exponent five, any solution to the Fermat Equation would have to be at least thirty digits long. Germain attempted to prove the following theorem that formalized her idea that any solutions to the Fermat Equation will be large enough to "frighten the imagination" [5]. This proof was also incomplete. However, Sophie Germain's Theorem, in which she is now famous for, came directly from her attempted proof of this Large Size Theorem [6].

Theorem (Large Size of Solutions) For the equation $x^p + y^p = z^p$ to be satisfied in whole numbers, p being any [odd] prime number, it is necessary that one of the numbers $x+y$, $z-y$, and $z-x$ be a muliple of the $(2p-1)^{th}$ power of the number p and of the p^{th} powers of all the prime numbers of the form $\theta = Np+1$, for which, at the same time, one cannot find two p^{th} power residues whose difference is one, and p is not a p^{th} power residue [6].

Germain also focused more on specific cases of exponents of the Fermat Equation. She formulated an attempt to prove Fermat's Last Theorem for exponent $2(8n \pm 3)$, and also for all even exponents, both incomplete.

No mathematician has ever proved that there exists an exponent in the Fermat Equation that has an infinite number of satisfying auxiliary primes. In fact, it has been proved that for certain integers, only a finite number of auxiliary primes exist. An undated letter to Legendre in which Germain proved that there are only finitely many auxiliary primes for $p = 3$ suggests that she realized her plan could not work for at least some exponents.

The grand plan cannot work for $p = 3$. For any prime θ of the form $6a + 1$, with $\theta > 13$. there are (nonzero) consecutive cubic residues. In other words, the only auxiliary primes for $p = 3$ for the non-consecutive condition are $\theta = 7$ and 13 [6].

6. Conclusion

These recent discoveries have shown that Sophie Germain deserves far more credit than previously given to her for her accomplishments in number theory. To this day, she is the only known woman to devote substantial time and effort to Fermat's Last Theorem and the first mathematician to formulate a plan to prove Fermat's Last Theorem in its entirety. She did all of this with no formal training or instruction and no mathematician to provide her with extensive feedback or correction.

Germain herself made strides so great to deserve the title of Princess of Mathematics. If more evidence is needed that this title is well earned, note that Gauss considered himself the Prince of

Mathematics and the subject of number theory in known as the Queen of Mathematics, so Germain's intense involvement with these two makes The Princess of Mathematics an even more suitable title. Her perseverance and passion for mathematics make her an inspiring mathematician to say the least.

References

[1] Alexanderson, Gerald L. "About the Cover: Sophie Germain and a Problem in Number Theory." Bulletin of the American Mathematical Society 49, no. 2 (2012): 327-31. doi:10.1090/s0273-0979-2012-01370-8.

[2] Centina, Andrea Del. "Letters of Sophie Germain Preserved in Florence." Historia Mathematica 32, no. 1 (2005): 60-75. doi:10.1016/j.hm.2003.11.001.

[3] Centina, Andrea Del. "Unpublished Manuscripts of Sophie Germain and a Revaluation of Her Work on Fermats Last Theorem." Archive for History of Exact Sciences 62, no. 4 (2007): 349-92. doi:10.1007/s00407-007-0016-4.

[4] "Fermat's Last Theorem." Wikipedia. September 09, 2018. Accessed September 09, 2018. https://en.wikipedia.org/wiki/Fermat's_Last_Theorem.

[5] Laubenbacher, Reinhard C., and David Pengelley. Mathematical Expeditions: Chronicles by the Explorers. New York: Springer, 2000.

[6] Laubenbacher, Reinhard, and David Pengelley. "Voici Ce Que Jai Trouv: Sophie Germains Grand Plan to Prove Fermats Last Theorem." Historia Mathematica 37, no. 4 (2010): 641-92. doi:10.1016/j.hm.2009.12.002.

[7] Musielak, Dora. Prime Mystery: The Life and Mathematics of Sophie Germain. Bloomington: AuthorHouse, 2015.

[8] Ribenboim, Paulo. Fermat's Last Equation for Amateurs. New York: Springer Verlag, 1999.

[9] "Sophie Germain and Fermat's Last Theorem." Sophie Germain and FLT. Accessed October 27, 2018. https://www.agnesscott.edu/lriddle/women/germain-FLT/SGandFLT.htm.