Introduction

"Three out of two people have trouble with fractions," is a statement that is intended as a joke, but unfortunately, trouble with fractions is not something to be taken lightly. The 2014-2015 academic year was my first year as a high school math teacher. I taught ninth through twelfth grade at a college preparatory private school in Augusta, Georgia. I witnessed firsthand just how much students struggle with fractions and how that lack of understanding can continue to haunt a student in his or her mathematical journey.

The students in my classes came from a variety of mathematical backgrounds and as a whole had attended several different elementary schools in the area. With the diversity of elementary teacher influence and an overwhelming number of students with fractional misconceptions, I began to wonder where was the disconnect. Were all of them exposed to the same type of learning environment in elementary school when they learned fractions? Had the teacher taught fractions in a different, more conceptual way, would my high school students still struggle with fractions?

In my undergraduate studies, I took a few math education classes with preservice teachers. Math education classes at Georgia College are designed to push conceptual understanding of future teachers in order to better their ability to teach students conceptually first followed by procedure. It was common to see my preservice classmates have difficulty with the more conceptual part of the class and a want to stay in their mathematical comfort zone of procedures.

In my study, I will analyze the preservice teachers' conceptual and procedural understanding of mixed number addition and subtraction. I wanted to look into preservice teachers' knowledge of fractions, specifically mixed number addition and subtraction, prior to being instructed on the topic at the collegiate level. I will also ask subjects about their personal relationships with math and their mathematical background. I have anticipated seeing a possible connection between the subjects' type of understanding with not only their mathematical background but also their personal relationship with math.

Literature Review

The National Council of Teachers of Mathematics has been asking for a reform in the math classroom since 1989. The Common Core is asking teachers to instruct in a way that balances both a conceptual understanding with procedural fluency. With current teachers struggling to accept the need for change, a lot of research has been done on preservice teacher education and knowledge in mathematics. I have reviewed this research along with how attitudes toward math and personal experience can affect performance. In my study, I observed the possible connection between the two.

Research on Preservice Teachers

"While mathematical knowledge for teaching has started to gain attention as an important concept in the mathematics teacher education research community, there is limited understanding of what it is, how one might recognize it, and how it might develop in the minds of teachers." (Silverman & Thomson, p. 499)

Before and After Fractional Knowledge

In Kristie Jones Newton's study, the participants were elementary education majors enrolled in an elementary mathematics education course. The course was specifically designed to strengthen knowledge in elementary mathematics. In Newton's study, eighty-five preservice teachers took both a pretest and posttest on fractional knowledge. She studied the existence of common misconceptions, subjects' preference to use algorithms, and mathematical flexibility. At the end of her study, Newton (2008) states,

> "For the most part, students in this study tended to use general algorithms even when they were not efficient, and it seems the course was not enough to alter the patterns of these students. Although students seemed to have fewer misconceptions at the end of the semester, their low flexibility suggests that they were still in the early stages of competence." (p. 1105)

What Newton is referring to as flexibility, is very similar to the conceptual understanding I studied. A main reason I wanted to look into my subjects' mathematical background and personal relationship with math was to gain a better understanding of why making conceptual connections can tend to be a point of struggle in preservice teachers.

Division of Fractions

In Deborah Loewenberg Ball's study, she analyzed a combination of preservice elementary and secondary mathematics teachers' knowledge of division of fractions. She studied both the computational skill level along with the level of understanding of the concept. While the majority of her subjects did successfully divide the fractions, few could explain the meaning of the division of fractions. In fact, most of the subjects admitted to only knowing the rule was to invert and multiply but had little to no understanding of why this procedure was effective.

Ball (1990) mentioned that a vast amount of math textbooks separate division of fractions from division of whole numbers completely. Hence, students commonly are not pushed to make connections between the two topics. Ball states, "These prospective teachers' responses to our questions were therefore not surprising, given the way in which they had probably been taught mathematics themselves." (p. 459) This is a great representation of the cycle of predominately procedural students becoming procedural teachers and in turn creating more procedural students.

Mathematical Knowledge for Teaching

"Shulman (1986, 1987) coined the term pedagogical content knowledge (PCK) to address what at that time had become increasingly evident—that content knowledge itself was not sufficient for teachers to be successful," (Silverman &Thomson, p. 409). More directly, requiring preservice elementary teachers to take more math classes will not in turn make them better math teachers. The term mathematical knowledge for teaching, or MKT, describes mathematical knowledge that is specifically useful in teaching mathematics. Silverman and Thompson discussed the lack and need of some sort of effective road map of developing MKT in preservice teachers. They describe the developing of MKT as transforming personal understanding of concepts into understanding not only how this understanding could influence student connections to later concepts but also, as a teacher, what he or she could do to support students' development in the concept and why or why not specific actions would work.

The idea of MKT is exactly what the Common Core is asking of teachers today. Common Core is pushing for teachers to help students make personal connections to concepts and wants teachers to encourage unique solutions. This can easily become overwhelming to a teacher without MKT.

Conceptual Understanding verses Procedural Fluency

In the book “Adding It Up”, the term mathematical proficiency is used to describe the combination needed to truly be successful in learning mathematics. Mathematical proficiency has five strands: conceptual understanding, procedural fluency, strategic competence, reasoning, and disposition. For my study, I chose to specifically focus on conceptual understanding and procedural fluency.

Conceptual Understanding

“Conceptual understanding refers to an integrated and functional grasp of mathematical ideas.” (Kilpatrick, Swafford, Findell, p. 118) This type of understanding can enable a student to learn at a faster pace because he or she is able to make connections from past concepts to the situation at hand due to an actual grasp of what the previous mathematical concepts truly represent rather than rote memorization of algorithms. Conceptual understanding also leads to retention of concepts. Algorithms are destined to be forgotten, but a deeper understanding leads to a greater chance of the student being able to manipulate their knowledge to either correct the mistake in the algorithm or solve the problem at hand in a completely different way.

Procedural Fluency

“Procedural fluency refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently.” (Kilpatrick, Swafford, Findell, p. 121) This is more than memorization of algorithms and timed math fact sheets. Procedural fluency is seeing that math is predictable, organized, and filled with patterns. Although making the connection is conceptual understanding, knowledge of the procedures and the ability to apply them is as equally crucial as making the deeper connection.

Developing Fraction Operations

In the math education classes at Georgia College, the required textbook is "Elementary and Middle School Mathematics, Teaching Developmentally" by VandeWalle. In this book, VandeWalle discusses strategies the future teachers might use when teaching fractional operations in their own classroom and common misconceptions.

Fractional Strategies

"Some teachers may argue that they can't or don't need to denote so much time to fraction operations, that sharing one algorithm is quicker and leads to less confusion of students." (VandeWalle, p. 372) However, fractional computational success has been related to success in Algebra 1. The idea of teaching students one specific way to do a problem leads to a complete lack of conceptual understanding which, as previously mentioned, leads to a lack of long term knowledge. VandeWalle suggested teaching fractional computation with this process:

- Use contextual tasks.
- Explore each operation with a variety of models.
- Let estimation and invented methods play a big role in developmental strategies.
- Address common misconceptions.

Although teaching in this way can be time consuming, overtime and with consistency, it is a good strategy to help develop both procedural fluency and conceptual understanding among students.

Common Misconceptions

"It is important to explicitly talk about common misconceptions with fraction operations because students naturally overgeneralize what they know about whole-number operations," (VandeWalle, p. 382). When adding and subtracting fractions, the most common error is to add both numerator and denominator. A major cause of this error is a lack of understanding in what the numerator and denominator actually represent. On the other end of the spectrum, the least common error is to not find a common denominator at all and completely ignore the denominators existence and add both numerators. This could imply the student doesn't understand what the denominator represents. Finding a common denominator is drilled into most students to the point they assume that is always necessary. However, knowing one needs a common denominator does not necessarily lead to knowing how to find one. Another common error is for students to not be able to quickly find the common multiples of the denominator.

Anxiety Towards Mathematics

Liisa Uusimaki & Rod Nason did a study the causes behind preservice teachers' negative beliefs about math and mathematical anxiety. Majority of subjects specifically recalled their negative feelings about math starting in elementary school. The root of subjects' anxiety pointed mostly towards their past elementary teachers. In studying preservice teachers myself, this article made me wonder how much of an impact a teacher's personal mathematical anxiety and negative associations with math can affect his or her students. Is this a never ending cycle of math anxiety being passed on from teacher to student over and over?

Situational Anxiety

The preservice teachers described two specific situations which caused the most math anxiety for them at the college level. Majority found verbally communicating their knowledge was the most stressful, as in, when asked in a math education to describe their work in front of the class or explain how to solve a problem. Another chronic cause of anxiety for the preservice teachers was teaching a lesson in math. Subjects were afraid of not using correct terminology, making a simple math error, and not being able to solve the problem. Anxiety in preservice teachers rooted in teaching mathematics when majority of them will eventually teach math is clearly a point of concern.

Types of Math

The types of math that caused the most anxiety were algebra and patterns (33%), space (31%), and number operations especially division (21%). These subjects primarily fall in the third through fifth grade range. One specific subject described being taught to memorize a song in order to remember her times tables. She recalled not being able to remember the song correctly and was never taught an alternate route of mental multiplication.

Methodology

I asked both of Dr. Abney's MATH 2008 classes to participate in an approximately fifteen minute long assessment in mid-October. Prior to the day of the assessment, possible participants were asked to complete a consent form to return on day of assessment in order to participate in study. MATH 2008 is Numbers and Operations course which is a prerequisite class at Georgia College and State University for preservice teachers to enter both the Special Education cohort and Early Childhood Education cohort. The class contains primarily freshmen and sophomore females. As a number and operations course, mixed number addition and subtraction will be covered at some point after my assessment. (See Appendix A for class syllabus.) This class was also chosen because this topic falls within the grade levels both sets of preservice teachers will be certified to teach upon graduation.

The assessment I designed was five questions directed towards analyzing conceptual verses procedural understanding of mixed number addition and subtraction, a concept taught in fifth grade under the Common Core curriculum. As the subjects are preservice teachers, I also wanted to study their ability to comprehend correct student work that differs from the way they personally prefer to solve the problems. As mentioned previously, there have been studies connecting problems with fractions to difficulties in Algebra 1 in high school. Part of this assessment was focused towards analyzing if this connection exists among these preservice teachers. I was also interested in learning what type of relationship the subjects had with mathematics in the sense of if it was a fond experience throughout school thus far or if it had always been a subject in which he or she struggled. Whether a negative or positive outlook, how, if at all, does that relate to his or her type of understanding. Calculators were not allowed during the assessment. (See Appendix B for assessment)

The first question simply asked subjects to solve a mixed number addition problem. The purpose of this question was to achieve an understanding of the natural instincts the preservice teachers have towards mixed number addition. Since the subjects come from a variety of mathematical backgrounds, I wanted to check for correctness first, along with the appearance of common misconceptions, and even conceptual or procedural understanding.

Next, I asked if they could solve the same problem in a different way. I gave a slight hint by suggesting a picture. This question was intended to push the subjects to think past his or her instinctual way of solving the problem and to encourage conceptual thinking rather than pure procedural fluency which I expected to see in the answers given to question one. I anticipated seeing several students not be able to solve the same problem with a different method. This could be from a lack of conceptual understanding or from a negative perspective of fractions in general limiting the subject from even trying to think conceptually about the problem. I will use the last two questions of the assessment to help determine the difference in the two.

The third question was three examples of student work. The subjects were asked to analyze each part individually and determine whether the work was always correct, sometimes correct, or incorrect. If answered sometimes correct or incorrect, the subjects were asked to please explain. A problem is only correct if both the work is mathematically correct and leads to the correct answer. Sometimes correct would be if the student obtains the right answer, but the work provided only leads to the correct answer in this specific case or a few specific cases but is not true for all cases. Incorrect would be when the work is either mathematically impossible or the answer is incorrect. All three student works were correct and were designed to be out of the norm. This was intended to study the preservice teachers' ability to not only recognize correct work that might be outside of their mathematical comfort zone but also to test how many of them might fall into the category of teachers who strictly encourage only the procedural aspect of mathematics. Different or unusual can at first sometimes seem incorrect to those whom depend primarily on procedures. Hence, if a subject chooses sometimes correct or incorrect, it could imply a discomfort with conceptual thinking of their own students. A lack of explanation behind a sometimes correct or incorrect response could also show a lack of personal conceptual thinking of the subject.

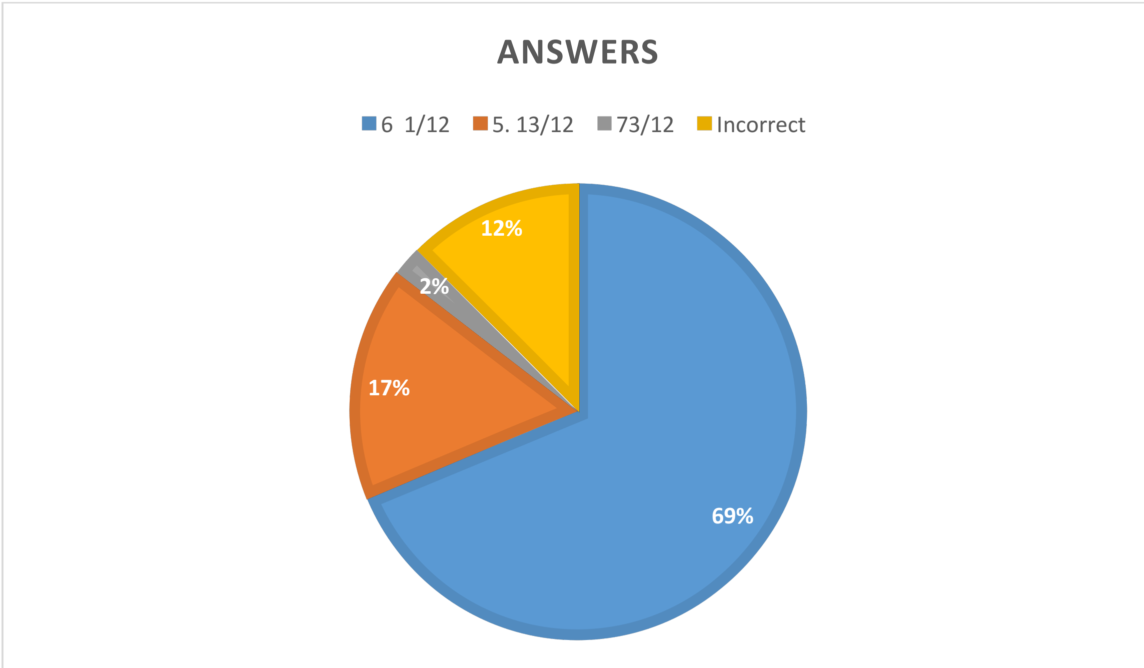
The next question was not a mixed number addition question. I asked if Algebra was an animal, what animal it would be. Subjects were asked to explain their answer. With this question, I wanted to get a better understanding of each subjects' mathematical background and personal relationship with math. During my research I became intrigued by the possible connection between overly procedural understandings of fractions with difficulties in Algebra 1. I did not want to merely ask if subjects were successful in Algebra because success wouldn't verify whether or not the struggle initially existed and had to be overcome.

Continuing to study the subjects' mathematical background and personal relationship with math, the final question asked subjects to complete a simile about learning mathematics. My goal was to see if each subject viewed mathematics as a conceptual field, procedural field, or some combination of the two. Since the subjects are preservice teachers how they view learning mathematics will directly relate to how they eventually teach mathematics in their own classroom. In asking this question, I was hoping to study if there might be a connection between the preservice teachers' personal relationship with math and their type of understanding of mixed number addition and subtraction. Unfortunately, even if a teacher has a conceptual understanding of a specific subject, he or she could still think teaching math is purely procedural.

Findings

Question 1

For question 1, I asked participants to solve $2\frac{3}{4} + 3\frac{1}{3}$. I scored question one for procedural fluency, or correctness. Correct answers received 1 point; incorrect received 0 points. Of the forty-eight subjects, 87.5% answered correctly. From the correct group, 19% did not simplify their answer but left an improper fraction with a whole number, $5\frac{13}{12}$. Participants who left their answer in the form $5\frac{13}{12}$ received 0.5 points. I chose to count these answers correct but consider it as an example of procedural rather than conceptual understanding of the problem. I was surprised only one student converted the mixed numbers to improper fractions before solving. One student left the problem blank stating, "I can't remember." The following pie chart represents the answers for question one.



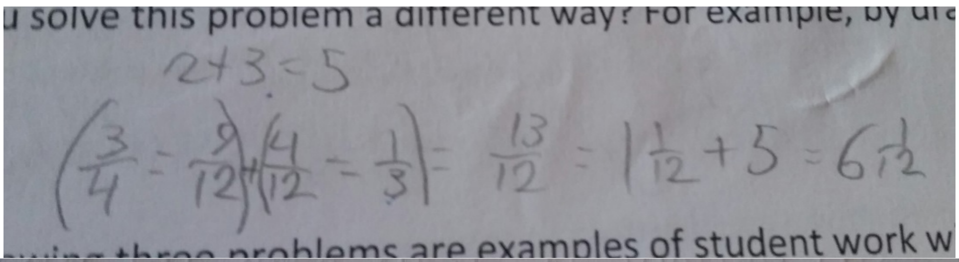
The most common error was to add the whole that comes from adding the fractions to the whole number but not reducing the improper fraction. The next most common mistake was an error in finding a common denominator when adding the fractions. One participant only added the fractions and ignored the whole numbers.

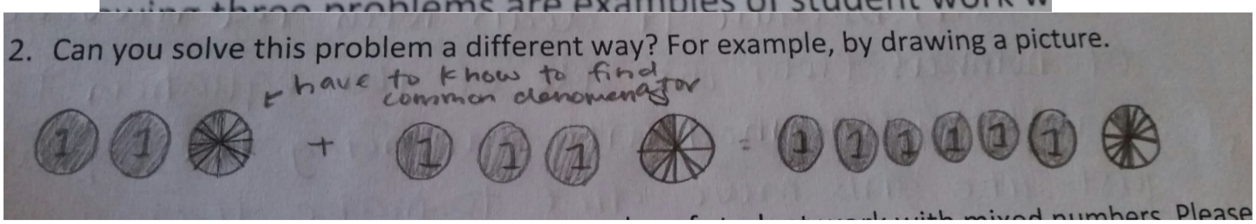
Question 2

Question 2 asked participants to solve the previous problem in a different way. I specifically focused on conceptual understanding for question two. The majority, (83%), of subjects could not accurately solve the problem in a different way. Out of those subjects, 43% did not attempt the problem. I counted no attempt as 0 points for conceptual understanding. If a subject was on the right track but could not finish completely or came to the wrong conclusion, he or she received a 0.5 for a somewhat conceptual understanding. The 17% who did correctly solve the problem in a different way received a 1.

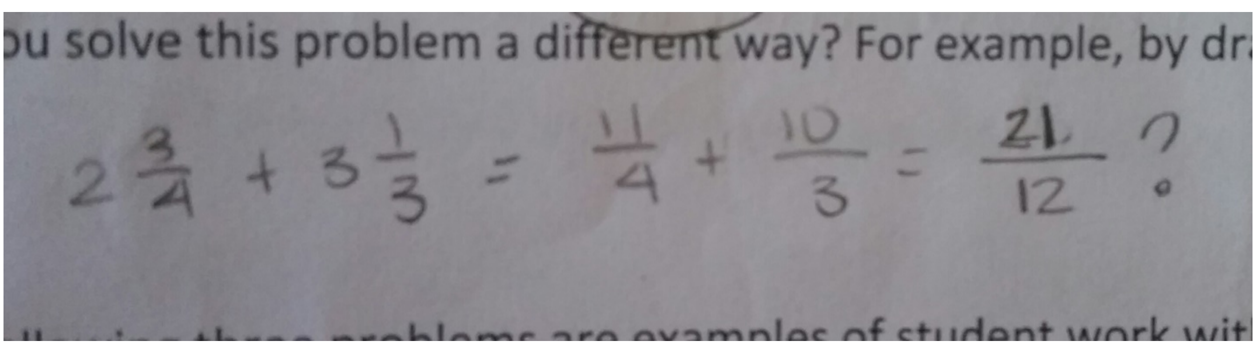
Since I suggested solving the problem again by drawing a picture, a vast amount of participants who did attempt the problem went this route. If in the drawing a subject did not represent how the fractions combined, I did not count the work correct. Of the 26 participants who attempted to solve the problem by drawing, 23% received full credit.

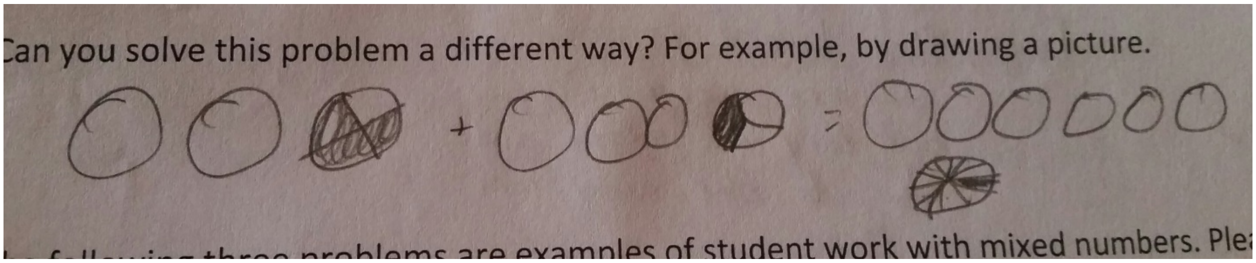
The following is an example of student work which received full credit.





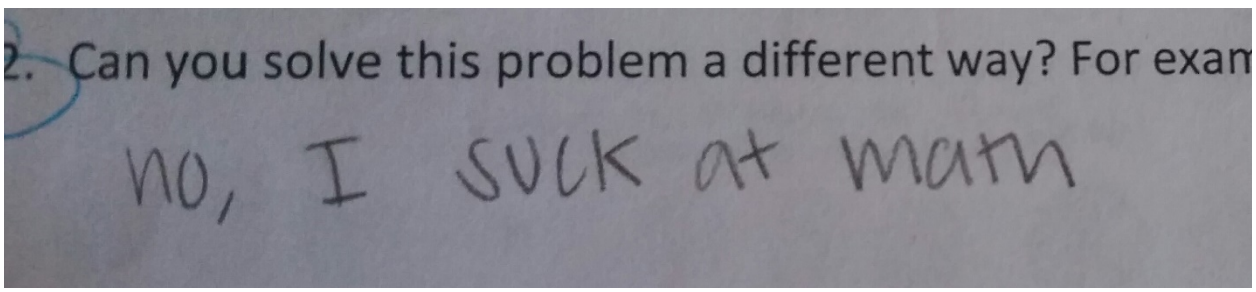
The following are examples of students who received 0.5 points.

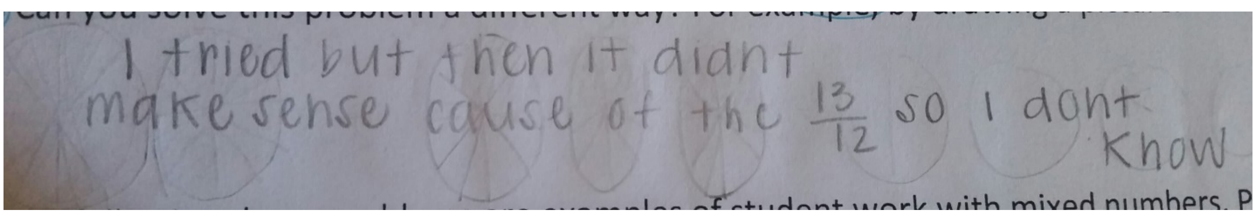




Note that, the first example correctly converted the mixed numbers to improper fractions and found the correct common denominator. However, the student did not adjust the numerators accordingly. The second example represent a common error I observed. The student drew pictures of the work done in question 1 instead using pictures to solve the problem again.

The following are examples of student work which received 0 points.





It is important to notice the erased work on the last example. The student attempted to solve the problem by drawing. The individual answered question 1 incorrectly as $6\frac{13}{12}$. When the student began to get a different answer, he or she assumed the process used in question must be correct.

Question 3

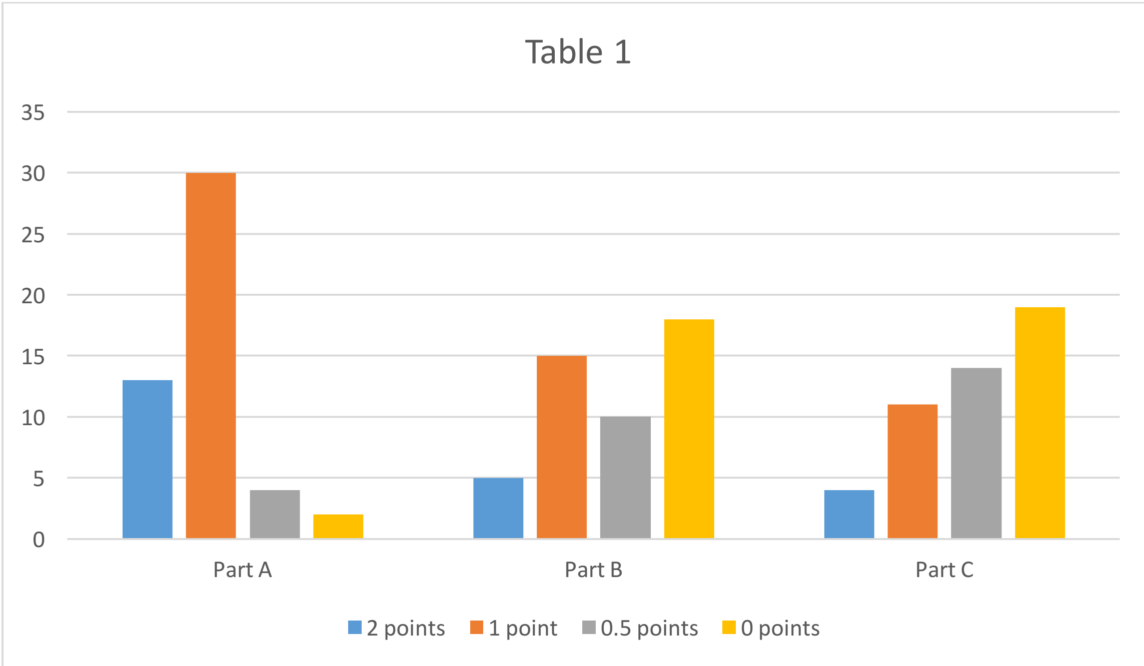
Participants were asked to analyze three examples of student work for question three and to explain their answers. Since for question three, subjects were asked to explain their answer, to receive a 2 a subject had to have both the correct answer and a correct explanation of choice. Correct answer without explanation yielded a 1, and an incorrect answer with explanation scored a 0.5. An incorrect answer with no explanation received 0 points. The following are examples of explanations that were not considered valid:

- This is how I do it.
- This is the answer I got.
- No, they didn't get the answer I got.
- Did not show enough work.

Part A. was answered correctly by 87.5% of subjects. However, of the correct answers, 31% also provided a correct explanation. Thus, only 27% of subjects received a two for conceptual understanding of the problem. On the other hand, 4% of subjects received 0 points.

Part B. was answered correctly by 42% of subjects. Only 10% of subjects also provided a valid explanation and received two points. Of the incorrect answers, 21% of subjects received a 0.5, and 37.5% of subjects received no points.

Part C. was the least correctly analyzed student work. Just 8% of the subjects received a two. Then, 23% received 1 point, 29% received 0.5 points, and 40% of subjects received 0 points. This student work had the least amount of steps in the solution. Several students thought the small amount of work implied incorrect. The following table represents the points per part in question 3.

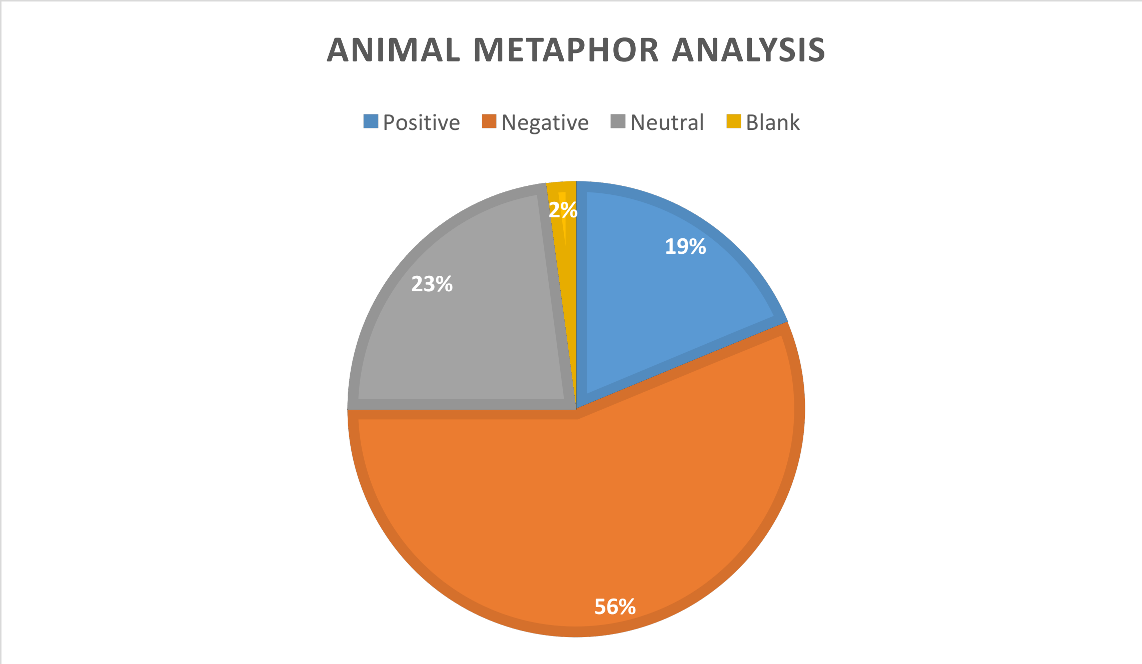


Question 4

Question 4 asked participants to compare Algebra to an animal and explain. No points were awarded in question 4. I analyzed participants' animal choice and explanation. Each response was placed in one of the following categories: positive, negative, and neutral. Here are some subject examples and the categories I placed them in:

- Positive- "It would be like a pit bull because it may seem scary at first but in reality it's normally harmless."
- Negative- "A cat because I don't like cats, and I don't like cats."
- Neutral- "It would be a fish because it is easy for some to get and difficult for others."

Note that, one subject left the question blank. The following pie chart represents the subjects' answers.



Question 5

For question 5, participants were asked to complete a simile about learning mathematics. Question five returned to the point system. The participants' responses to the simile was determined to be a procedural comparison or conceptual. One point was awarded for conceptual; zero points were awarded for a procedural choice. For question five, subjects were also asked to explain their answer. After analyzing the explanations, all answers fell into three categories: conceptual, procedural, or mathematical relationship. I considered a response conceptual if a student referred to math in a large scheme or referred to how all things in math are intertwined together. A procedural explanation contained any mention of steps or algorithms. Some participants related their choice to their personal mathematical relationship, or feelings about math. The following are examples of each category:

Conceptual:

Everything you learn throughout your school years in math builds off of each other.-what you learn one year is dependent on your understanding of what you learned the year before.

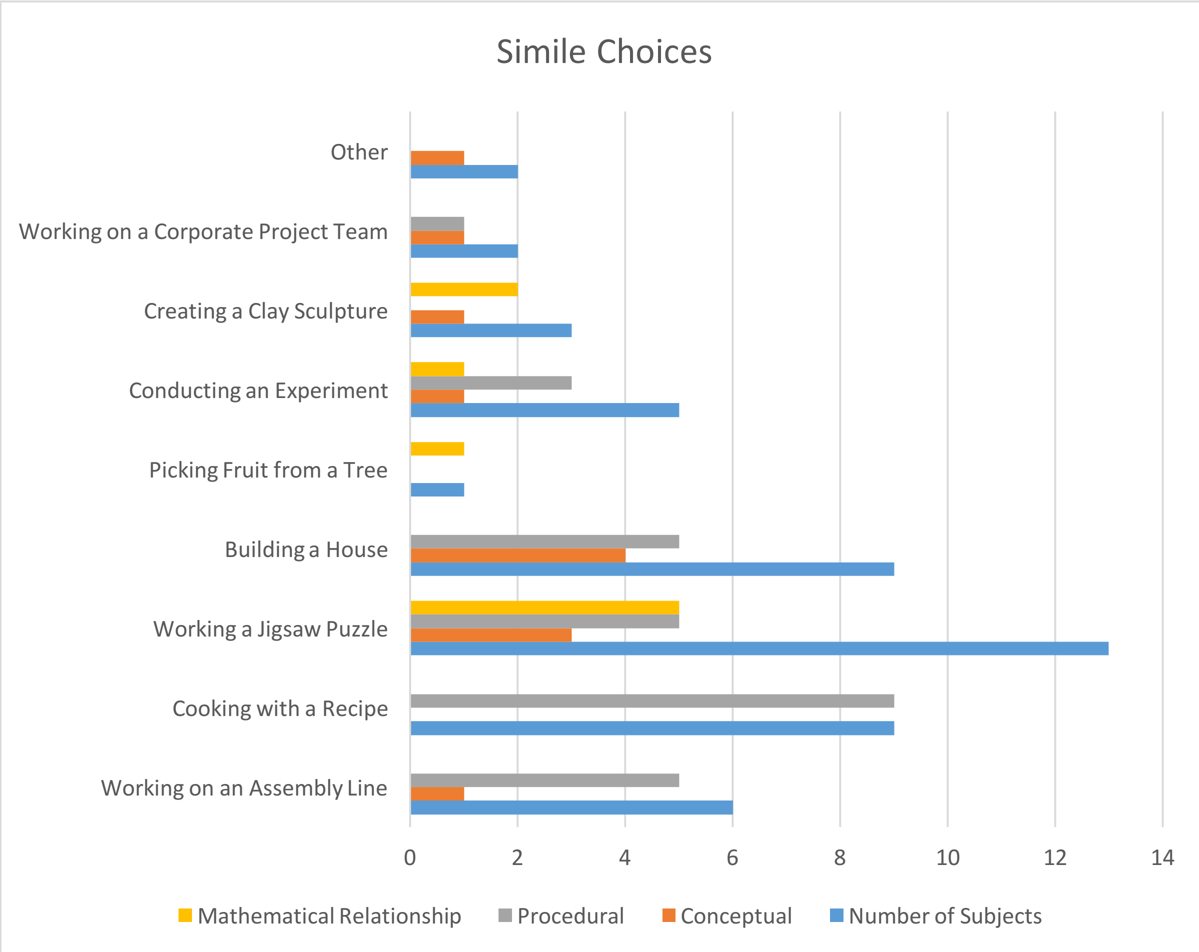
Procedural:

Each type of problem has a specific formula, just like a recipe, so if you follow it correctly you will get the right results.

Mathematical Relationship:

Jigsaw puzzles are not easy. People feel lost and confused doing them. That's how I feel about math.

The chart Simile Choices represents subjects' simile answers along with explanations.



The total procedural response was 58%. The next greatest response was 25% conceptual, followed by 16% mathematical relationship.

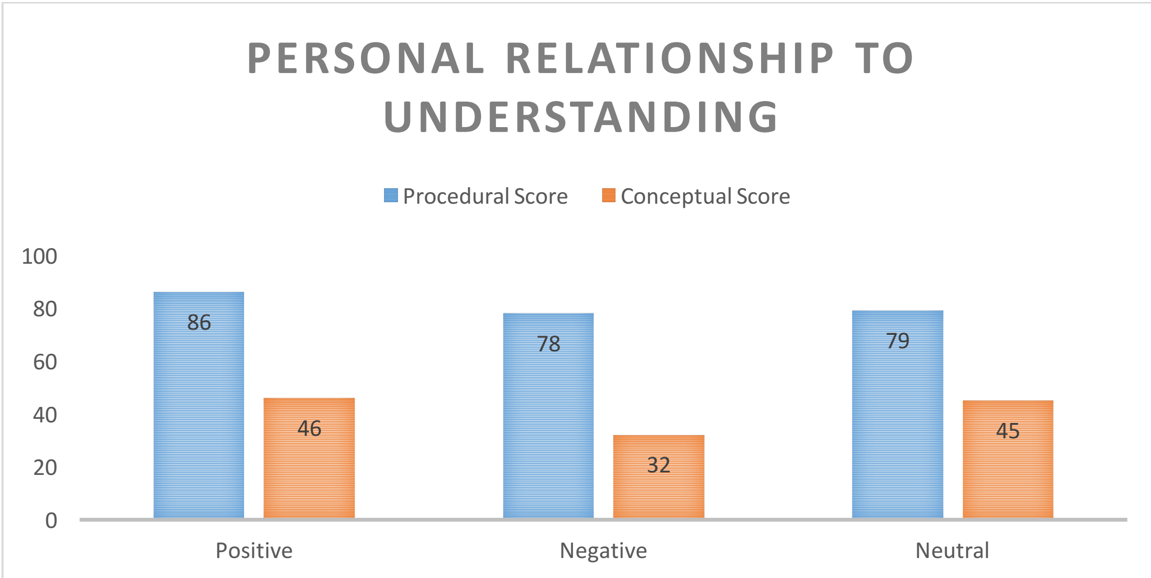
Scoring

Each student received both a procedural score and a conceptual score. The procedural scores were determined from question 1 only and range from 0 to 1. The conceptual scores were determined from questions 2 and 3 ranging from 0 to 7. The following table shows each student's procedural score, conceptual score, and mathematical relationship.

Participant	Procedural Score 0 to 1	Conceptual Score 0 to 7	Mathematical Relationship (+,-, neutral)
Student 1	1	3	Neutral
Student 2	1	4	+
Student 3	1	3.5	+
Student 4	0	2	Neutral
Student 5	1	6.5	Neutral
Student 6	1	2.5	-
Student 7	.5	2.5	Neutral
Student 8	1	1.5	-
Student 9	1	1.5	-
Student 10	1	1	-
Student 11	1	2.5	-
Student 12	1	3	-
Student 13	1	2.5	-
Student 14	1	1	-
Student 15	1	3	-
Student 16	1	2.5	-
Student 17	1	2.5	+
Student 18	.5	2.5	-
Student 19	1	1	-
Student 20	.5	1	-
Student 21	1	2	-
Student 22	1	2	-
Student 23	1	6	-
Student 24	1	1	-
Student 25	1	1	-
Student 26	.5	3	-
Student 27	1	1	Neutral
Student 28	1	3	+

Student 29	.5	3	-
Student 30	0	3	-
Student 31	0	3.5	Neutral
Student 32	1	4	Neutral
Student 33	1	2	-
Student 34	1	1.5	Neutral
Student 35	1	5.5	-
Student 36	1	4	+
Student 37	1	2	-
Student 38	1	2.5	Neutral
Student 39	0	1	-
Student 40	0	5	-
Student 41	0	2	+
Student 42	1	3.5	+
Student 43	1	2	Neutral
Student 44	.5	2	-
Student 45	1	4	Neutral
Student 46	1	5.5	Neutral
Student 47	.5	.5	-
Student 48	.5	1	-

From the data, I separated my conceptual and procedural findings into personal mathematical relationship categories. The following chart displays my findings.



Of my participants, I can conclude there is a connection between their personal relationship with math and the type of understanding they have of mixed number addition and subtraction. Note that, scores were converted to percentages in order to have a comparable scale. Students with a positive mathematical relationship also had the highest conceptual score and procedural score followed by neutral relationship and then negative.

Conclusions

The National Council of Teachers of Mathematics has been calling for mathematical reform since 1989. More than twenty-five years later, a true major change cannot be observed in the average math classroom. In my research, I found that preservice teachers have a good procedural understanding of addition and subtraction of mixed numbers. However, they lack a conceptual understanding. In addition, the preservice teachers who had a positive personal relationship with math tended to have a greater conceptual understanding when compared to those with a negative personal relationship with math. The following topics are branches for future research in relation to “The Mystery of Mixed Number Addition and Subtraction.”

The Common Denominator Epidemic

From my research and experience teaching, I observed students tend to primarily remember finding a common denominator and associate the process with most fractional operations. When analyzing the student work on the assessment, I had several students mark the work incorrect because no common denominator was found. It is safe to assume in elementary school, the students were drilled with repetition to find a common denominator when adding or subtracting fractions. Is pushing this process limiting conceptual knowledge of later fractional operations?

Common Core Conflict

As mentioned previously, the Common Core is currently asking teachers to teach in a way that balances procedural fluency with conceptual understanding. Some teachers feel teaching in this way is too time consuming and only confusing the students more. If teachers are going to refuse to reform, what does that mean for the future of American mathematics?

Procedural Cycle

With the push back against the Common Core, most students are still learning primarily procedurally. These procedural students are then becoming procedural preservice teachers. While in college, these procedural preservice teachers are being taught to teacher with a balance of procedural and conceptual, but unfortunately, most will teach within their comfort zone of mostly procedural. Thus, they will in turn make more procedural students. What will it take to disrupt the procedural cycle and find the balance?

Appendix A

Appendix B