Quipus and Their Influence Seen Through Mathematical Analysis

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Fall 2012

It is difficult to understand a culture when not much evidence survives, or when we fail to have a full understanding of the clues left behind. The Incas, a culture that flourished for 100 years yet was overcome in a mere 30 (Ascher, M and Ascher, R.), remain a mystery to the modern researcher in many ways. It is easy to interpret them based on a Western mentality, which may or may not give the correct interpretation. "Written accounts are distorted as they pass through this route: one culture (Inca) is interpreted via a second culture (Spanish), which is interpreted via a third culture (American)" (Ascher, M. and Ascher, R.). Sometimes, however, it is not writing that is left behind but, rather, and object. Problems arise when that object does not have a counterpart in another culture (Ascher, M. and Ascher, R.). Such problem arises from the existence of Quipus. Quipus can be described as "knotted-string devices that were used for bureaucratic recording and communication" by the Incas (Urton and Brezine). Cords with knots are not the most obvious way to begin to explain a culture, but quipus may hold the true complexity of the mathematical understanding of the Incas.

The structure of a quipu gets complicated very fast. The base of each individual quipu begins in the same manner, but the body becomes more complex as the matter being recorded gains complexity. The building blocks of a quipu are the cords which are individually made and then assembled. These individual cords that are constructed, when amassed into a quipu, each have a specific function. There is a main cord serves as a basis; it is "generally much thicker than the pendant cords" (Ascher, M. and Ascher, R.). The main cord expresses horizontal direction in a quipu. The cords that are tied to the main cord are called pendant cords. When laid behind the main cord, pendant cords point downward. If the cords point up, they are called top cords. Because of the pendant and top cords, a quipu is said to have vertical direction; pendant cords go down and top cords go up. A special cord that can be attached to the looped end of the main cord is referred to as a dangle end cord. There can also be cords that are not directly attached to the main cord; these are referred to as subsidiaries. Within each quipu, the cords are placed in a hierarchy: first comes the main cord, then the pendant cords, then the subsidiary cords. A quipu, therefore, would look like this:

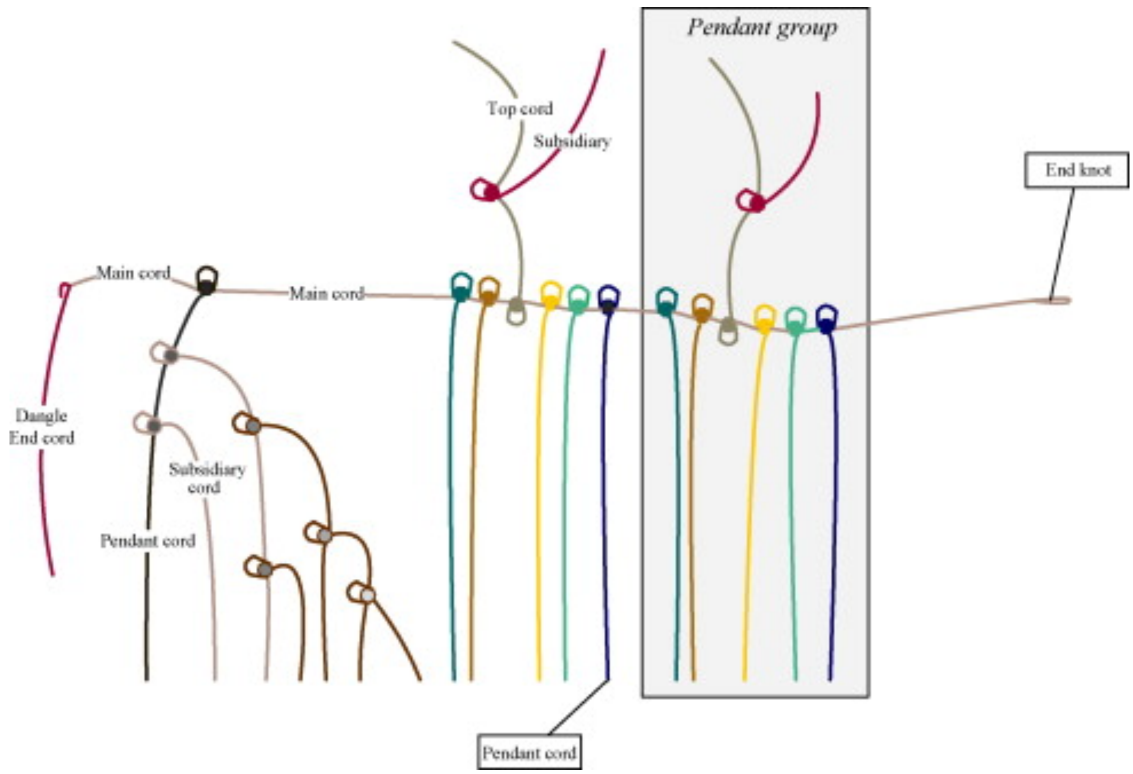

Figure 1. Structure of a quipu. (Beynon-Davies)

Notice in the figure above that the cords on the quipu vary in color and in spacing. This is where the differences between individual quipus come into play. Notice the difference in color and spacing shown in figure 1. These differences were intentional; they merely showed different categories of what was being recorded in the quipu.

Color and spacing are important for the separation of levels and/or categories. The concept of using color to identify or set something apart has been implemented in modern society, but the color schemes used by the quipumakers are not as easily relatable or decipherable. Color and color patterns displayed in a quipu are important, and their importance shows in the fact that all quipus found display intricate patterns to express the information encoded in a quipu. A centralized meaning for color combinations and patterns does not exist. Colors used within a quipu have an exclusive meaning to the person who made the specific quipu. The pattern formed by the colors being used depicts the relationship being expressed (Ascher M. and Ascher, R.). Once again, this pattern is exclusive to the individual quipumaker. The lack of uniformity of meaning for color made it mandatory that each individual quipu be interpreted only by those who made it (Julien). Because of the exclusivity of color and patterns, the important idea to take away from this is that it is the relationship between the cords that matters the most, not the individual cords themselves.

Another differentiating characteristic of each quipu can be noted in the content of the cords. Each individual cord was filled with different knots to represent a number. These knots, however, were only representational since they "have become abstracted from their association with particular object" (Ascher, M. and Ascher, R.). The knots tell us how many, but they do not express what they record.

The quipumakers utilized three different types of knots to express value. There are long knots, single knots, and figure eight knots. Single knots can be seen in any of the other positions to represent a value between one and nine. Figure eight knots are used in the units position to represent a value of 1 because (Ascher, M. and Ascher, R.). The long knots are used in the units position to represent a value between two and nine. The units position can hold two types of knots because the long knot can be made with a minimum of two loops. Figure 2 shows the three different types of knots and what they look like.

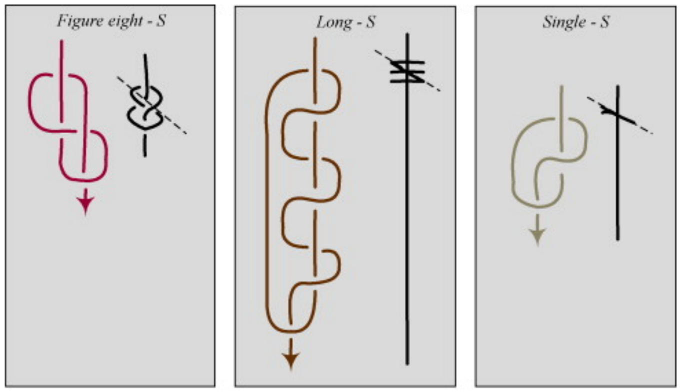

Figure 2. Types of knots (Beynon-Davies, 2009)

These individual knots are placed in clusters of no more than nine individual knots, just like a base ten decimal system (Ascher, M and Ascher, R.). The position of the clusters is very important because it determines the value of that cluster. As was mentioned before, each type of knot can only be used in certain value positions. As the clusters move closer to the main cord, their value increases by a power of ten. This means that the very last cluster has a power of ten of zero; the next one has a power of one, then a power of two and so on. It must be noted that top cords are read in the opposite direction.

The next logical question to ask is: how did the Incas represent the concept of 0? Because the Incas used a base 10 decimal system, it was very important that they have an understanding of the notion of the value 0. There are three major parts that make up the concept of 0: 1.) understanding the idea that a position with no individual value adds to the overall value, 2.) nothing must be represented in some way or another, and 3.) when the representation of nothing stands alone, it is to be considered a number (Ascher, M. and Ascher, R.). Quipus record a value of 0 in two ways: by leaving a pendant cord blank or by using neighboring cords as a reference.

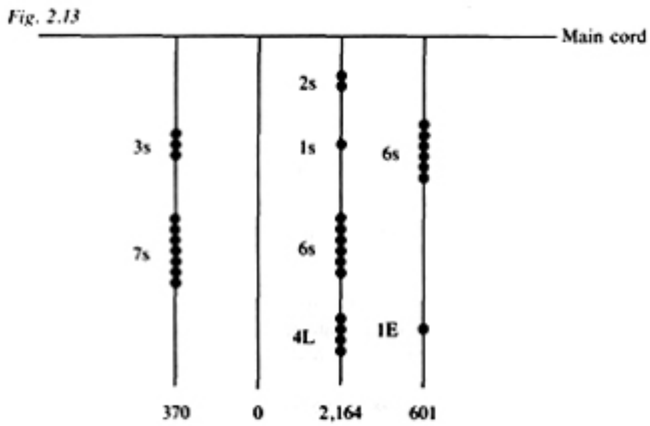

Figure 3. Concept of 0 (Ascher, M. and Ascher, R.)

Both of these ways are demonstrated in figure 3. In this figure, S represents a single knot, L represents a long knot and E represents a figure-eight knot. The second pendant cord is left blank, indicating 0. Notice that the position of the clusters on each cord are level to each other to make it easier to note when a position is left empty, as in the fourth pendant cord.

It is important to now note that some of the cords with clusters of knots are summation, or summary, cords. This means that these particular cords register the sums of numbers corresponding to a group of pendant cords (Pereyra).

The basic structure of a quipu has been described, but what type of information can be stored on such an object? Quipus were used by the Inca Empire as a way of keeping records, a way of accounting. In order to prevent records on a quipu from being altered, duplicate quipus were made and given to another party (Julien). This is known because almost identical quipus have been found time and time again, suggesting the idea of duplicate quipus. Each quipumaker was in charge of a region, denoted as levels, and he had to keep records of the activities for that level.

In order to understand the responsibilities of the Inca accountants, we must first understand their role in society and the division of the Inca Empire. At the lowest level of the hierarchy was the Chunka Kamayoq, meaning organizer of 10, who oversaw 10 tributaries. The next level up was composed of 5 groups of 10, overlooked by a Pichoa-Chunka Kuraka. Climbing up the hierarchy level, the Pachaka Kuraka was in charge of 2 groups of 50. The next level up was the head of each of the provinces, the T'oqrikoq. Above the T'oqrikoq were the lords of each of the 4 quarters into which Cuzco was divided. At the very top of the hierarchy was the Inca King in Cuzco (Urton and Brezine). This hierarchy facilitated communication and structure from one level of power to another, ensuring the prosperity of the empire.

We can now begin to explore the information stored in the quipus that was to be passed from the lower levels up and vice versa. Quipus could record countless types of information. They held information about tributaries and their tributes, the results of the harvests, the population, and even the days of the year.

The analysis of quipus suggests arithmetical ideas that are both physically and not physically expressed in the quipus. Let us first explore the ideas that are evident on a quipu itself. Division into equal parts can be seen by the partition of the groups. This partition can only be done with integers, even if that means dividing a value like 45 into 22 and 23. (Ascher, M and Ascher, R.). For example if a quipu has a total value of 60, then it will be expressed by two groups of 30 and one of these two groups will be expressed by two groups of 15. Division into parts can also be seen within the individual knot clusters. For example, a cluster of value two can be divided into two knots of value one. This idea of dividing the knots can be compared to the distributive axiom (Ascher, M and Ascher, R.).

The relationship that the values recorded have with each other is also important in giving more information about the arithmetical ideas implemented in the quipus. Before analyzing this relationship, note that decimal fractions will be used to express this relationship. This is because decimal fractions take away the importance of the actual integers being compared and place the importance on the relation that forms from the comparison (Ascher, M and Ascher, R.).

We begin the analysis by looking at quipu AS120. This quipu consists of four groups of eight pendant cords each, with 32 total pendant cords. The table form of the quipu can be seen in figure 4. From figure 4 we can observe the following relationship: $G_1=G_2+G_3+G_4$, where G_i represents a group (Pereyra S.). Adding up the columns will result in the value given by G_1 in that particular column. For example, for the first column: 25691=8731+11730+5230. This suggests a summation property within a quipu. In other words, one group represents the sum of all of the other groups.

G_1	25691	1068	42760	1896	19274	5485	6541	43372
G_2	8731	362	14743	641	6521	1862	2222	14701
G_3	11730	457	18053	808	8220	2330	2777	18475
G_4	5230	249	9964	448	4533	1293	1542	10196

Figure 4. AS120 Groups and values. (Pereyra S.)

Another characteristic that can be seen on AS120 is that dividing any G_i will result in an almost constant decimal value. This can be seen in figure 5. Comparing G_2/G_1, an almost constant value of 0.340 appears; G_3/G1's ratio is 0.425. The ratio of G_4/G_1 is 0.235. All three of these ratios are expressed in each situation to within one percent (Ascher, M. and Ascher, R.). The consistency of these values suggests the idea that the Incas were capable of manipulating numbers by constant values. Somehow, the values of AS120 may have been manipulated by any of the three ratios mentioned, although no physical evidence of this manipulation is present on the quipu. With further analysis of AS120, we can see that the quotient relationship can be expanded to all of the groups.

G_2/G_1	0.340	0.339	0.345	0.338	0.338	0.339	0.340	0.339
G_3/G_1	0.457	0.428	0.422	0.426	0.426	0.425	0.425	0.426
G_4/G_1	0.204	0.233	0.233	0.236	0.235	0.236	0.236	0.235

Figure 5. AS120 Quotients. (Pereyra S.)

We now analyze quipu AS143. This quipu is separated into two sections of cords. The first part has five groups of four or five cords each. The second one has four groups of two cords each. The tabular form of AS143 can be seen in figure 6. For G_1 through G_5, note that $G_1=G_2+G_3+G_4+G_5$, where G_i represents a group. Take column one for example: 40036=4412+9094+17436+9094+136+67+23+3. Like in AS120, a summation property is present. The quotients of the groups of AS143 can also be analyzed as with AS120. From figure 7 we can see that the division of any group G_i by G_1 gives an almost constant ratio. Again, this suggests that the Incas were capable of manipulation by constant decimal ratios. Quipu AS143 strengthens the arithmetical ideas present in AS120.

AS120 and AS143 have similar properties, suggesting that these are not a result of coincidence. These quipus suggest three important conclusions: there is a summation property within a quipu, manipulation of numbers by constant decimal ratios, and indication of quantitative information.

G_1	40036	18504	97357	1350	23098
G_2	4412	2026	10856		2511
G_3	9094	4156	22469		5343
G_4	17436	8220	41883	1350	9528
G_5	9094	4102	22149		5716
G_6	136	106			
G_7	67	49			
G_8	23	76			
G_9	3	1020			

Figure 6. AS143 setup. (Pereyra S.)

G_2/G_1	0.110	0.109	0.112		0.109
G_3/G_1	0.227	0.225	0.231		0.231
G_4/G_1	0.436	0.444	0.430		0.413
G_5/G_1	0.227	0.222	0.228		0.247

Figure 7. AS143 Quotients (Pereyra S.)

The next quipu we will look at is AS55&AS56. This quipu consists of two small quipus combined, possible only because the two were discovered together. One quipu consists of seven pendant cords and the other of three pendants (Ascher, M. and Ascher, R.). Ignoring the first pendant cord of the first quipu, a tabular representation of is given by figure 8. On this figure, P_{ij} represents the pendant from group i, cord j where i=1,2,3 and j=1,2,3. When analyzing quipu AS55&AS56, Ascher and Ascher made two very interesting observations:

$$\frac{P_{11}P_{12}P_{13}}{P_{31}P_{32}P_{33}} = \frac{P_{31}P_{32}P_{33}}{P_{21}P_{22}P_{23}}$$

$$P_{32}P_{32}P_{33} = P_{11}P_{22}P_{33}$$

The first equality says that the product of the values of the third row of figure 8 is equal to the geometric mean of the product of the values of the first and second row. The second equality states that the product of the values of the third row is equal to the product of the values in the diagonal (Ascher, M. and Ascher, R.). These two observations make quipu AS55&AS56 interesting from an arithmetic perspective.

P_{11}	P_{12}	P_{13}
P_{21}	P_{22}	P_{23}
P_{31}	P_{32}	P_{33}

Figure 8. AS55 & AS56 tabular form (Ascher, M and Ascher, R.)

This quipu also shows signs of cyclical arithmetic being performed, as can be seen in figure 9. The ratios that are present in the table are ratios that showed up over and over when the original values of the quipu were compared. The values on rows two and three can be expressed as by the values on row one and a combination of the ratios found. This cyclical relationship that is observed on AS55&AS56 suggests that these calculations may have been intentional (Ascher, M. and Ascher, R.). It suggests that the Incas were capable of multiplication of integers by fractions.

P_{11}	P_{12}	P_{13}
$\left(\frac{11}{14}\right)\left(\frac{7}{8}\right)P_{11}$	$\left(\frac{7}{8}\right)\left(\frac{34}{33}\right)P_{12}$	$\left(\frac{34}{33}\right)\left(\frac{11}{14}\right)P_{13}$
$\left(\frac{7}{8}\right)P_{11}$	$\left(\frac{34}{33}\right)P_{12}$	$\left(\frac{11}{14}\right)P_{13}$

Figure 9. Another look (Ascher, M and Ascher, R.)

There are also quipus that were not used to store quantities, but calendric years. The pachaquipu, quipu of time, suggests that the Incas were capable of keeping time in a western world's standards (Lauraencich-Minelli and Magli). A graphic of the pachaquipu can be seen in figure 10. This calendar, found in the *Exsul Immeritus Blas Valera Populo Suo,* has a representation for twelve months and a total of 365 days. The pachaquipu remains under investigation and analysis.

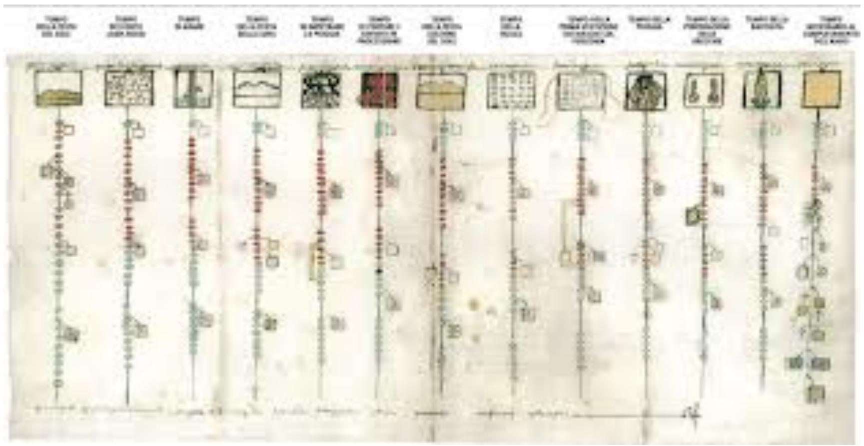

Figure 7. Pachaquipu (Laurencich-Minelli and Magli)

In conclusion, quipus held information vital for the understanding and prosperity of the Incan Empire. Quipus held key data that helped the empire run smoothly; they kept almost exact records of taxpayers, censuses, and supplies available. Quipus can be considered among the most intricate structures not involving written language. Quipus suggest that the Incas had a more complex understanding of mathematics than physical evidence show. Various relationships of values not visible can be discovered through analysis of quipus. In the very words of Martha and Robert Ascher, "To the very minimum, their knowledge included addition, division into equal parts, division into simple unequal fractional parts, division into proportional parts, multiplication of integers by integers and multiplication of integers by fractions."

Literature Cited

Ascher, M. and Ascher,R. *Mathematics of the Incas: Code of the Quipu*. Dover Publications, Inc. New York. 1981.

Beynon-Davies, P. 2009. Significant threads: The nature of data. *International Journal of Information Management.* 29(3): 170-188.

Julien, C. J. 1988. How Inca Decimal Administration Worked. *Ethnohistory*. 35(3): 257-279. http://www.jstor.org/stable/481802

Laurencich-Minelli, L and Magli, G. 2009. A calendar Quipu of the early 17th century and its relationship with the Inca astronomy. *Archaeoastronomy.* 22. http://arxiv.org/ftp/arxiv/papers/0801/0801.1577.pdf

Pereyra, S. 1996. Acerca de dos quipus con características numéricas excepcionales. *Bulletin de l'Institut francais d'études Andins.* 25(2):203-231.

Urton, G. and Brezine, C. J. 2005. Khipu Accounting in Ancient Peru. *Science*. 309: 1065-1067.