Fractional Calculus Fundamentals and Applications in Economic Modeling

by

Austin McTier

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Dr. Jebessa Mijena, Assistant Professor of Mathematics

Abstract

A relatively untapped branch of calculus, Fractional Calculus deals with integral and differential operators of non-integer order, as well as resolving differential equations consisting of said operators. This paper examines certain properties, definitions, and examples of fractional integrals, Riemann-Liouville fractional derivatives, Caputo fractional derivatives and differential equations, along with various methods in order to solve them. In addition, this paper applies a fractional order approach to modeling the growth of the economies of the United States and Italy, particularly their gross domestic products (GDPs). Based on previous research, we expect to find that the implemented fractional models will have a stronger performance than alternative methods of measuring economic growth.

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Chapter 1

Introduction

In the late seventeenth century, Leibniz established the notation $\frac{d^n}{dx^n}f(x)$, which denotes the nth derivative of a function f, with the implication that $n \in \mathbb{N}$. When this was reported to de l'Hospital, de l'Hospital responded by questioning the significance of the operator if $n = \frac{1}{2}$ [3]. The specific questioning of Leibniz's operator in regards to $n = \frac{1}{2}$, a fraction, gave rise to the labeling of this branch of mathematics as Fractional Calculus, though n need not be restricted to $\mathbb{Q}$; in fact, for this paper, $n \in \mathbb{R}$ applies to all operators in the following text (n may also apply to $\mathbb{C}$, though we will only go in depth for $\mathbb{R}$) [3].

While there exist many generalizations for solving derivatives and integrals of non-integer order, we will only be analyzing one method for fractional integrals (Riemann Liouville Integrals), and two methods for solving fractional derivatives, the Riemann-Liouville derivative and the Caputo derivative. It should be noted that while the Riemann Liouville derivative was historically the first (developed in the former half of the nineteenth century), the Caputo derivative is a more appropriate method when dealing with pragmatic problems [3]; this will be discussed in more detail later on. Operators of fractional order can also be used to solve ordinary differential equations of fractional order, as will be discussed in detail later. While fractional calculus has existed as a theoretical branch of math since it's integer ordered counterpart, its use in practical applications was sparse at best until the past few decades when a rather large number of scientific branches, such as physics, chemistry, finance, and engineering, began applying fractional differential equations to problems in said fields [3]. As such, this paper analyzes the particular application of fractional calculus in order to make better economic growth model predictions than traditional classical calculus based methods.

Chapter 2

Fractional Calculus: Definitions and Examples

2.0.1 Fractional Integrals

First, let $n \in \mathbb{R}$. We define the Riemann - Liouville fractional integral operator of order n as

$$K_a^n f(t) = \frac{1}{\Gamma(n)} \int_\alpha^t (t-u)^{n-1} f(u) du,$$

for $a \leq t \leq b$, where $f : [a, b] \to \mathbb{R}$, f is measurable on $[a, b]$ and $\int_a^b |f(t)| dt < \infty$ (which we define as the Lebesgue space $L_p[a, b]$ where p = 1) [3, definition 2.1]. Note that, for $n = 0, K_a^0$ = I, the identity operator. By the definition, it is apparent that the Riemann-Liouville integral concurs with the classical definition of the integral for $n \in \mathbb{N}$, with the exception that we have extended the domain of n to $\mathbb{R}$.

Example 2.1. Let $f(t) = t^p, p > -1$. Then

$$\begin{aligned} K_0^n t^p &= \frac{1}{\Gamma(n)} \int_0^t (t-u)^{n-1} u^p du \\ &= \frac{t^{n+p} B(p+1, n)}{\Gamma(n)} = \frac{\Gamma(n+1)}{\Gamma(n+p+1)} t^{p+n} \end{aligned}$$

where $B(x, y) = \displaystyle\int_0^1 u^{x-1}(1-u)^{y-1} du$ is the beta function. For example,

$$K_0^{0.5} t^2 = \frac{\Gamma(0.5+1)}{\Gamma(0.5+2+1)} t^{2.5} = \frac{t^{2.5}}{3.75}$$

Note that when $n = 1$, we obtain

$$K_0^1 t^p = \frac{\Gamma(1+1)}{\Gamma(1+p+1)} t^{p+1} = \frac{1}{p+1} t^{p+1}$$

Example 2.2. Let $f(t) = e^{\lambda t}$. Then

$$\begin{aligned}
K_0^n e^{\lambda t} &= \frac{1}{\Gamma(n)} \int_0^t (t-u)^{n-1} e^{\lambda u} du \\
&= \frac{1}{\Gamma(n)} \int_0^t x^{n-1} e^{\lambda(t-x)} dx, \text{ where } x = t-u, \\
&= \frac{e^{\lambda t}}{\Gamma(n)} \int_0^t x^{n-1} e^{-\lambda x} dx \\
&= \frac{e^{\lambda t}}{\Gamma(n)} \int_0^{\lambda t} \frac{p^{n-1}}{\lambda^{n-1}} e^{-p} \frac{1}{\lambda} dp, \text{ where } p = \lambda x, \\
&= \frac{e^{\lambda t}}{\lambda^n \Gamma(n)} \int_0^{\lambda t} p^{n-1} e^{-p} dp \\
&= \frac{e^{\lambda t} \Gamma^*(n, \lambda t)}{\lambda^n \Gamma(n)}, \qquad (2.1)
\end{aligned}$$

where

$$\Gamma^*(s, x) = \int_0^x p^{s-1} e^{-p} dp$$

is lower incomplete gamma function.

For special case, $n = 1$, we get the integer order integral

$$K_0^1 e^{\lambda t} = \int_0^t e^{\lambda p} dp = \frac{e^{\lambda t}}{\lambda \Gamma(1)} \Gamma^*(1, \lambda t) = \frac{e^{\lambda t} - 1}{\lambda}.$$

2.0.2 Properties of Riemann-Liouville Integrals and Fractional Derivatives

Next, we define the generator form of a fractional derivative $\frac{d^\alpha f(x)}{dx^\alpha}$ as

$$\frac{d^\alpha f(x)}{dx^\alpha} = \int_0^\infty [f(x) - f(x-y)] \frac{\alpha}{\Gamma(1-\alpha)} y^{-\alpha-1} dy,$$

[2, Pg. 30]. By applying integration by parts to the generator form, where $z = f(x) - f(x-y)$, we obtain both the Caputo form, defined as

$$\mathbb{D}_0^\alpha f(x) = \tfrac{1}{\Gamma(1-\alpha)} \int_0^\infty \frac{d}{dx} f(x-y) y^{-\alpha} dy,$$

and the Riemann-Liouville form, defined as,

$$D_0^\alpha f(x) = \tfrac{1}{\Gamma(1-\alpha)} \tfrac{d}{dx} \int_0^\infty f(x-y) y^{-\alpha} dy$$

[2, Pg. 30]. Note that we may substitute $z = x - y$ in either the Caputo or Riemann-Liouville form to obtain the integrand in the form $f(z)(x-z)$.

Example 2.3. Let $f(t) = e^{\lambda t}$ for some $\lambda > 0$ such that $f'(t) = \lambda e^{\lambda t}$. By substituting $x = \lambda$u, it follows that the Caputo derivative is

$$\begin{aligned}
\mathbb{D}_0^\alpha e^{\lambda t} &= \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \frac{d}{dt} e^{\lambda(t-u)} u^{-\alpha} du \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \lambda e^{\lambda(t-u)} u^{-\alpha} du \\
&= \frac{\lambda e^{\lambda t}}{\Gamma(1-\alpha)} \int_0^\infty e^{\lambda(-u)} u^{-\alpha} du \\
&= \frac{\lambda e^{\lambda t}}{\Gamma(1-\alpha)} \int_0^\infty e^{-x} \left(\frac{x}{\lambda}\right)^{-\alpha} \frac{dx}{\lambda} \\
&= \frac{\lambda e^{\lambda t}}{\Gamma(1-\alpha)} \lambda^{\alpha-1} \int_0^\infty e^{-x} x^{(1-\alpha)-1} dx \\
&= \frac{\lambda e^{\lambda t}}{\Gamma(1-\alpha)} \lambda^{\alpha-1} \Gamma(1-\alpha) = \lambda^\alpha e^{\lambda t}. \qquad (2.2)
\end{aligned}$$

This agrees with the integer case, for example

$$\mathbb{D}_0^3 e^{\lambda t} = \lambda^3 e^{\lambda t}.$$

Similarly, the Riemann-Liouville derivative is

$$\begin{aligned}
D_0^\alpha e^{\lambda t} &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^\infty e^{\lambda(t-u)} u^{-\alpha} du \\
&= \frac{d}{dt} \left(\frac{e^{\lambda t}}{\Gamma(1-\alpha)} \int_0^\infty e^{\lambda(-u)} u^{-\alpha} du \right) \\
&= \frac{d}{dt} \left(\frac{e^{\lambda t}}{\Gamma(1-\alpha)} \lambda^{\alpha-1} \Gamma(1-\alpha) \right) \\
&= \frac{d}{dt} (\lambda^{\alpha-1} e^{\lambda t}) = \lambda^\alpha e^{\lambda t}. \qquad (2.3)
\end{aligned}$$

Example 2.4. Let $f(t) = \sin(ct)$ for $t \geq 0, c \in \mathbb{R}$. Note that the fractional derivative of $\sin(ct)$ (as well as $\cos(ct)$) can be solved using the aforementioned fractional derivative $\frac{d^\alpha}{dt^\alpha} e^{\lambda t}$ $= \lambda^\alpha e^{\lambda t}$. By Euler's formula, $e^{ict} = \cos(ct) + i \sin(ct)$, and that for $i \in \mathbb{C}, i^n = e^{\frac{i\pi n}{2}}$. It follows that the Caputo fractional derivative is

$$\begin{aligned}
\mathbb{D}_0^\alpha e^{ict} &= (ic)^\alpha e^{ict} = c^\alpha e^{\frac{i\pi\alpha}{2}} e^{ict} = c^\alpha e^{i(ct + \frac{\pi\alpha}{2})} \\
&= c^\alpha \cos(ct + \frac{\pi\alpha}{2}) + c^\alpha i \sin(ct + \frac{\pi\alpha}{2})] \\
&= \mathbb{D}_0^\alpha \cos(ct) + i \mathbb{D}_0^\alpha sin(ct). \qquad (2.4)
\end{aligned}$$

Hence $\mathbb{D}_0^\alpha \sin(ct) = c^\alpha \sin(ct + \frac{\pi\alpha}{2})$ and $\mathbb{D}_0^\alpha \cos(ct) = c^\alpha \cos(ct + \frac{\pi\alpha}{2})$.

Example 2.5. Let $f(t) = t^n, n > 0$, for $t \geq 0$ and $f(t) = 0$ for $t < 0$. Note that $f'(t) = nt^{n-1}$ for $t \geq 0$. Then the Caputo derivative for $f(t)$ is

$$\begin{aligned}
\mathbb{D}_0^\alpha f(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \frac{d}{dt} u^n (t-u)^{-\alpha} du \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{d}{dt} u^n (t-u)^{-\alpha} du \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t n u^{n-1} (t-u)^{-\alpha} du \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t n u^{n-1} (t-u)^{-\alpha} du \\
&= \frac{n}{\Gamma(1-\alpha)} \int_0^t u^{n-1} (t-u)^{(1-\alpha)-1} du \\
&= \frac{n}{\Gamma(1-\alpha)} \frac{\Gamma(n)\Gamma(1-\alpha)}{\Gamma(n+1-\alpha)} t^{n+(1-\alpha)-1} \\
&= \frac{n\Gamma(n)}{\Gamma(n+1-\alpha)} t^{n-\alpha} = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} t^{n-\alpha} \qquad (2.5)
\end{aligned}$$

It agrees with the integer case, for example

$$\mathbb{D}_0^2 t^4 = \frac{\Gamma(4+1)}{\Gamma(4+1-2)} t^{4-2} = 12t^2.$$

It follows that the Riemann-Liouville derivative is

$$\begin{aligned}
D_0^\alpha t^n &= \frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha)} \int_0^t u^n (t-u)^{-\alpha} du\right) \\
&= \frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha)} \int_0^t u^{(n+1)-1} (t-u)^{(1-\alpha)-1} du\right) \\
&= \frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha)} \frac{\Gamma(n+1)\Gamma(1-\alpha)}{\Gamma(n+2-\alpha)} t^{n+1-\alpha}\right) \\
&= \frac{\Gamma(n+1)}{\Gamma(n+2-\alpha)} (n+1-\alpha) t^{n-\alpha} = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} t^{n-\alpha} \qquad (2.6)
\end{aligned}$$

Note that the Riemann-Liouville form and the Caputo form may not always agree:

Example 2.6. Let $f(t) = 1$ for $t \geq 0$ and $f(t) = 0$ for $t < 0$. Then $f'(t) = 0$ for $t \neq 0$, thus it follows that the Caputo fractional derivative is zero as well. In fact, if $f(t) = c$ where

$c \in \mathbb{R}$, the Caputo derivative will always equal zero.

The same cannot be said for the Riemann-Liouville derivative. For $f(t) = 1, t > 0$ and $0 < \alpha < 1$, and substituting $x = t - u$,

$$\begin{aligned} D_0^\alpha f(t) &= \frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha)}\int_0^t (1)(t-u)^{-\alpha}du\right) \\ &= \frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha)}\int_0^t x^{-\alpha}dx\right) \\ &= \frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha)}\frac{t^{1-\alpha}}{1-\alpha}\right) = \frac{t^{-\alpha}}{1-\alpha} \neq 0. \end{aligned} \tag{2.7}$$

This inequality is the reasoning for the Riemann-Liouville derivative, while established well in terms of mathematical theory, proving problematic when applied to practical problems. In order to avoid said difficulties, the Caputo derivative was formed [3]. This reasoning will be fully recognized later when fractional differential equations are introduced.

2.0.3 Properties of Riemann-Liouville Integrals and Fractional Derivatives

Theorem 2.7. *[3, Theorem 2.1] Let $f \in L_1[a,b]$ and $n > 0$. Then $K_a^n f(t)$ exists for almost every $t \in [a,b]$. Furthermore, the function $K_a^n\ f$ is itself also an element of $L_1[a,b]$.*

Theorem 2.8. *[3, Theorem 2.2, Corollary 2.3] Let $m, n \geq 0$ and g is a function such that $g \in L_1[a,b]$. Then*

$$K_a^m K_a^n g = K_a^{m+n} g = K_a^{n+m} g = K_a^n K_a^m g,$$

holds almost everywhere on $[a,b]$. Additionally, if $m + n \geq 1$, then the identity holds everywhere on $[a,b]$.

Theorem 2.9. *[3, Theorem 2.13, Lemma 3.13] Let $n_1, n_2 \geq$, and let $h \in L_1[a,b]$ and $f = K_a^{n_1+n_2} h$. Then*

$$D_a^{n_1} D_a^{n_2} f = D_a^{n_1+n_2} f.$$

Similarly, for the Caputo Derivative, let f be a function such that $f \in f : [a, b] \to \mathbb{R}; f$ has a continuous kth derivative for some $a < b$ and some $k \in \mathbb{N}$. Furthermore, let $n, \delta > 0$ be such that there exists some $j \in \mathbb{N}$ with $j \leq k$ and $n, n + j \in [j - 1, j]$. Then

$$\mathbb{D}_a^{\delta} \mathbb{D}_a^n f = \mathbb{D}_a^{n+\delta} f.$$

Theorem 2.10. *[3, Theorem 2.14, Theorem 3.7]. Let $n \geq 0$. Then, for every $f \in L_1[a, b]$,*

$$D_a^n K_a^n f = f.$$

Similarly, if f is continuous and $n \geq 0$, then

$$\mathbb{D}_a^n K_a^n f = f.$$

Note that Theorem 2.10 states that the Riemann-Liouville and Caputo derivatives are left inverses of the Riemann-Liouville fractional integral for some function f. It follows, however, that neither are the right inverse of the Riemann-Liouville fractional for f. In the case of the Caputo derivative:

Theorem 2.11. *[3, Theorem, 3.8] Let $n \geq 0$ and $m = \lceil n \rceil$. Also let the Riemann-Liouville Fractional derivative D_a^n exist for some function f, where f possesses $m - 1$ derivatives at a. Then*

$$K_a^n \mathbb{D}_a^n f(t) = f(t) - \sum_{k=0}^{m-1} \frac{D^k f(a)}{\Gamma(k+1)} (t - a)^k$$

Theorem 2.12. *[3, Theorem, 2.17.] Let f and g be two functions defined on [a,b] st $D_a^n f$ and $D_a^n g$ exist almost everywhere. Furthermore, let $c, d \in \mathbb{R}$. Then, $D_a^n(cf + dg)$ exists almost everywhere, and*

$$D_a^n(cf + dg) = D_a^n cf + D_a^n dg = cD_a^n f + dD_a^n g.$$

Note that this property also holds true for the Caputo derivative.

Chapter 3

Fractional Differential Equations

For many ordinary differential equations, the Laplace Transform is an essential tool in order to determine solutions. For fractional ordinary differential equations, this is no different. We define the Laplace transform for function $f(t)$ is

$$F(s) = \mathcal{L}(f(t), s) = \int_0^\infty e^{-st} f(t)dt.$$

For $0 < \alpha < 1$, the Riemann-Liouville fractional derivative $D_0^\alpha f(t)$ has Laplace Transform $s^\alpha F(s)$. The Caputo derivative $\mathbb{D}_0^\alpha f(t)$ for $0 < \alpha < 1$, however, has Laplace Transform $s^\alpha F(s) - s^{\alpha-1} f(0)$. In fact, for $n < \alpha < n+1, n, n+1 \in \mathbb{N}$, the Caputo fractional derivative $\mathbb{D}_0^\alpha f(t)$ has a Laplace Transform $s^\alpha F(s) - s^{\alpha-1} f(0) + s^{\alpha-2} f'(0) + \cdots + s^{\alpha-(n+1)} f^{(n)}(0)$ [2].

Example 3.1. Let $f(t) = t^n, n > 0$, for $t \geq 0$ and $f(t) = 0$ for $t < 0, 0 < \alpha < 1$, and let $u = st$. Note that, using the definition of the Gamma function,

$$F(s) = \int_0^\infty e^{-st} t^n dt = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \frac{du}{s} = s^{-(n+1)} \int_0^\infty e^{-u} u^{(n+1)-1} du = s^{-n-1}\Gamma(n+1)\ .$$

Thus, the Caputo derivative $\mathbb{D}_0^\alpha f(t)$ has Laplace Transform

$$\int_0^\infty e^{-st} \mathbb{D}_0^\alpha f(t) dt = s^{\alpha-n-1}\Gamma(n+1) = [s^{-(n-\alpha)-1}\Gamma(n-\alpha+1)] \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}.$$

Inverting the Laplace Transform results in

$$\mathbb{D}_0^n t^n = \frac{\Gamma(n+1) t^{n-\alpha}}{\Gamma(n-\alpha+1)}.$$

Before we move forward to fractional differential equations, the establishment of Mittag-Leffler functions is needed. The elementary case $E_\beta(t)$ is defined, whenever the series converges, as the Mittag-Leffler function of x of order β

$$E_\beta(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\beta k + 1)}.$$

Note the special cases

$$E_0(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(1)} = \frac{1}{1-t} \text{ for } |t| < 1, \quad E_1(t) = e^t, \quad \text{and } E_2(t) = \cosh(\sqrt{t}).$$

The more general set of functions is defined as such. Let $\beta, \gamma > 0$. Then the function $E_{\beta,\gamma}(t)$ is defined, whenever the series converges, as the two-parameter Mittag-Leffler function with parameters β and γ such that

$$E_{\beta,\gamma}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\beta k + \gamma)}.$$

Note the special case $\gamma = 1$, which results in the previous Mittag-Leffler function of order β. Let $\beta, \gamma, \delta > 0$ and $\omega \in \mathbb{R}$. Then the function $E^\delta_{\beta,\gamma}(\omega t)$ is defined, whenever the series converges, as the three-parameter Mittag-Leffler function with parameters β and γ of degree δ such that

$$E^\delta_{\beta,\gamma}(\omega t) = \sum_{k=0}^{\infty} \frac{(\delta)_k (\omega t)^k}{k!\Gamma(\beta k + \gamma)},$$

where $(\delta)_k = (\delta)(\delta + 1)...(\delta + k - 1)$. Note that $(1)_k =$ k!.

It follows that if the Laplace transform of some function $f(t)$ is $F(s) = s^\gamma(1 - \omega s^{-\beta})^{-\delta}$, then the inverse Laplace transform of $F(s)$ is

$$f(t) = \mathcal{L}^{-1}(F(s), t) = t^{\gamma-1} E^\delta_{\beta,\gamma}(\omega t^\beta) = t^{\gamma-1} \sum_{k=0}^{\infty} \frac{(\delta)_k (\omega t^\beta)^k}{k!\Gamma(\beta k + \gamma)}, \tag{3.1}$$

where β, γ, $\delta > 0$ and $\omega \in \mathbb{R}$.

Additionally, the convolution is a transformation that is needed, particularly one case demonstrated later. For two functions $f(t)$ and $g(t)$ such that $f, g : [0, \infty) \to \mathbb{R}$, the convolution is defined as

$$(f * g)(t) = \int_0^t f(u)g(t-u)du.$$

Example 3.2. Let $f(t) = \sin(at)$ for $t \geq 0$, a > 0 and $0 < \alpha < 1$. Note that $f(0) = \sin(0) = 0$. It follows that $F(s) = \frac{a}{s^2+a^2}$. So the Caputo fractional derivative has Laplace Transform

$$s^\alpha F(s) - s^{\alpha-1}f(0) = \frac{as^\alpha}{s^2+a^2} = \frac{as^\alpha}{s^2(1+a^2s^{-2}} = as^{-(2-\alpha)}(1+a^2s^{-2})^{-1}.$$

Note that using equation (3.1), γ = 2 - α, ω = -a^2, β = 2, and δ = 1. Thus the inverse Laplace transform is

$$\begin{aligned} \mathbb{D}_0^n f(t) &= a\mathcal{L}^{-1}(s^{-(2-\alpha)}(1+a^2s^{-2})^{-1}, t) = at^{2-\alpha-1}E^1_{2,2-\alpha}(-a^2t^2) \\ &= at^{1-\alpha}\sum_{k=0}^{\infty}\frac{(-a^2t^2)^k}{\Gamma(2k+2-\alpha)} = at^{1-\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k(at)^{2k}}{(2k+1-\alpha)!}. \end{aligned} \tag{3.2}$$

If $\alpha = 1$, then $\mathbb{D}_0^1 f(t) = \frac{d}{dt}\sin(at) = at^{1-1}\sum_{k=0}^{\infty}\frac{(-1)^k(at)^{2k}}{(2k+1-1)!} = a\sum_{k=0}^{\infty}\frac{(-1)^k(at)^{2k}}{(2k)!} = a\cos(at)$.

3.1 Fractional Differential Equation

Theorem 3.3. *Let* $0 < \alpha < 1$. *The solution of the initial value problem*

$$\mathbb{D}_0^\alpha y + y = 0, \quad y(0) = y_0$$

is given by

$$y(t) = y_0 E_\alpha(-t^\alpha).$$

Proof. Taking Laplace transform of the differential equation we have

$$s^\alpha F(s) - s^{\alpha-1}y_0 + F(s) = 0.$$

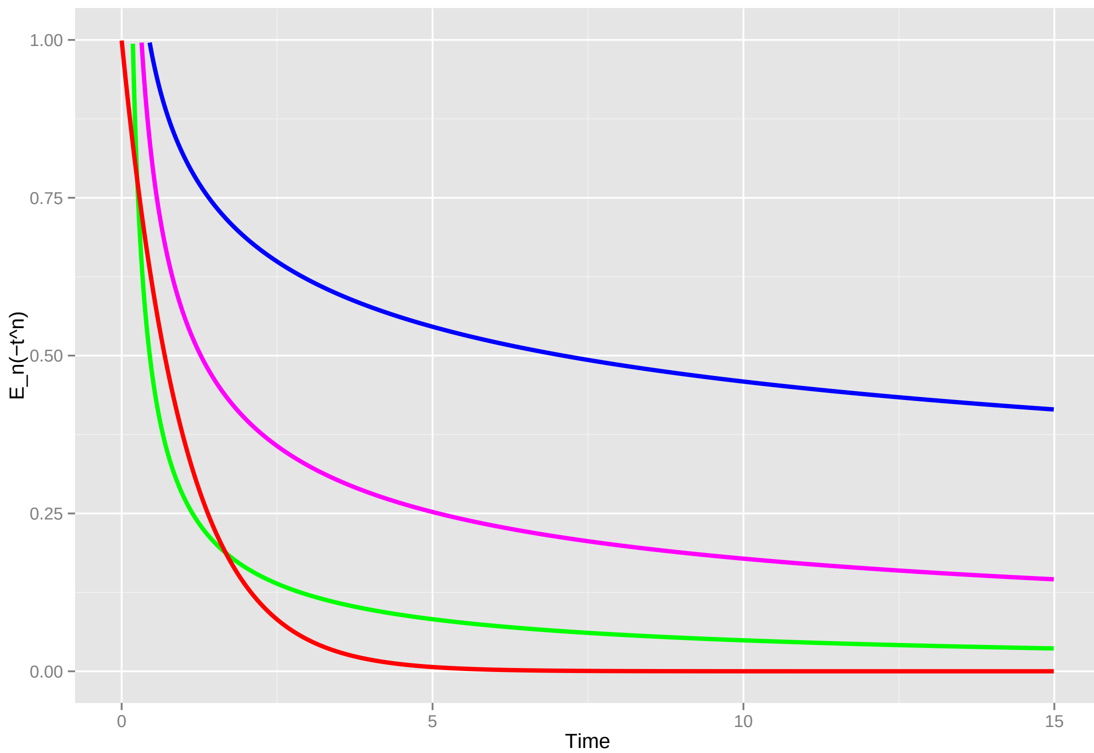

Figure 3.1: Plots of $E_\beta(-t^\beta)$ for $\beta = 1/4$, $\beta = 1/2$, $\beta = 3/4$, and $\beta = 1$

This implies,

$$F(s) = \frac{s^{\alpha-1}y_0}{s^\alpha + 1} = y_0[s^{-1}(1 + s^{-\alpha})^{-1}].$$

Using equation (3.1) the solution to the differential equation is

$$y(t) = y_0 t^{1-1} E^1_{\alpha,1}(-t^\alpha) = y_0 E_\alpha(-t^\alpha) \tag{3.3}$$

If $\alpha = 1$, then

$$y(t) = y_0 \sum_{k=0}^{\infty} \frac{(-1)^k t^{\alpha k}}{(\alpha k)!} = y_0 \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} = y_0 e^{-t},$$

which agrees with the solution of the integer case $y'(t) + y(t) = 0, y(0) = y_0$. $\square$

Theorem 3.4. *Let $0 < \alpha < 1, 1 < \beta < 2$. Then the solution of the initial value problem*

$$\mathbb{D}_0^\beta y + \mathbb{D}_0^\alpha y = 0 \quad y(0) = y_0, y'(0) = y_1,$$

is

$$y(t) = y_0 E_{\beta-\alpha}(-t^{\beta-\alpha}) + y_1 t E_{\beta-\alpha,2}(-t^{\beta-\alpha}) + y_0 t^{\beta-\alpha} E_{\beta-\alpha,\beta+1-\alpha}(-t^{\beta-\alpha})$$

Proof. The Laplace transform of this equation is

$$s^\beta F(s) - s^{\beta-1} y_0 - s^{\beta-2} y_1 + s^\alpha F(s) - s^{a-1} y_1 = 0$$
$$F(s)[s^\beta + s^\alpha] = s^{\beta-1} y_0 + s^{\beta-2} y_1 + s^{a-1} y_0$$

It follows that

$$\begin{aligned} F(s) &= \frac{s^{\beta-1} y_0}{s^\beta + s^\alpha} + \frac{s^{\beta-2} y_1}{s^\beta + s^\alpha} + \frac{s^{\alpha-1} y_0}{s^\beta + s^\alpha} \\ &= y_0 s^{-1} (1 + s^{-(\beta-\alpha)})^{-1} + y_1 s^{-2} (1 + s^{-(\beta-\alpha)})^{-1} \\ &\quad + y_0 s^{\alpha-1-\beta} (1 + s^{-(\beta-\alpha)})^{-1} \end{aligned}$$

So the Inverse Laplace Transform of $F(s)$ is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}(F(s); t) = y_0 t^{1-1} E^1_{\beta-\alpha,1}(-t^{\beta-\alpha}) \\ &+ \; y_1 t^{2-1} E^1_{\beta-\alpha,2}(-t^{\beta-\alpha}) + y_0 t^{\beta-\alpha} E^1_{\beta-\alpha,\beta+1-\alpha}(-t^{\beta-\alpha}) \end{aligned}$$

If $\beta = 2$ and $\alpha = 1$, the solution is

$$y(t) = y_0 \sum_{j=0}^{\infty} \frac{(-1)^j (t)^j}{j!} + y_1 \sum_{j=0}^{\infty} \frac{(-1)^j (t)^{j+1}}{(j+1)!} + y_0 \sum_{j=0}^{\infty} \frac{(-1)^j (t)^{j+1}}{(j+1)!}$$
$$= y_0 e^{-t} - [y_0 + y_1] e^{-t} + [y_0 + y_1] = -y_1 e^{-t} + [y_0 + y_1],$$

which agrees with the solution for

$$y'' + y' = 0,\ y(0) = y_0, y'(0) = y_1.$$

□

Theorem 3.5. *Let $0 < \alpha < 1$. Then the fractional differential equation*

$$\mathbb{D}_0^\alpha y + y = \sin(t),\ y(0) = y_0$$

has the solution

$$y(t) = \sin(t) * t^{\alpha-1} E_{\alpha,\alpha}(-t^{-\alpha}) + y_0 E_\alpha(-t^{-\alpha})$$

Proof. The Laplace transform of the differential equation is

$$s^\alpha F(s) - s^{\alpha-1} y_0 + F(s) = \frac{1}{s^2+1}$$

Then

$$\begin{aligned} F(s) &= \frac{1}{(s^2+1)(s^\alpha+1)} + \frac{s^{\alpha-1}y_0}{1+s^\alpha} \\ &= \left(\frac{1}{s^2+1}\right)\left(\frac{1}{s^\alpha+1}\right) + \frac{s^{\alpha-1}y_0}{1+s^\alpha}. \end{aligned}$$

So the inverse Laplace transform is

$$y(t) = \mathcal{L}^{-1}(F(s);t) \quad = \quad \sin(t) * t^{\alpha-1} E^1_{\alpha,\alpha}(-t^{-\alpha}) + y_0 E^1_{\alpha,1}(-t^{-\alpha})$$

It follows that when $\alpha = 1$,

$$\begin{aligned}
y(t) &= \sin(t) * t^{\alpha-1}\sum_{k=0}^{\infty}\frac{(-1)^k t^{\alpha k}}{\Gamma(\alpha k+\alpha)} + y_0\sum_{k=0}^{\infty}\frac{(-1)^k t^{\alpha k}}{\Gamma(\alpha k+1)} \\
&= sin(t) * \sum_{k=0}^{\infty}\frac{(-1)^k t^k}{\Gamma(k+1)} + y_0\sum_{k=0}^{\infty}\frac{(-1)^k t^k}{\Gamma(k+1)} \\
&= sin(t) * e^{-t} + y_0 e^{-t} \\
&= \int_0^t sin(u)e^{-(t-u)}du + y_0 e^{-t} \\
&= \int_0^t sin(u)e^{-t}e^{u}du + y_0 e^{-t} \\
&= e^{-t}\int_0^t sin(u)e^{u}du + y_0 e^{-t}
\end{aligned}$$

Using the integral

$$\int \sin(u)e^u du = e^t\left(\frac{\sin(t)}{2} - \frac{\cos(t)}{2}\right) + \frac{1}{2}$$

Hence

$$y(t) = e^{-t}\int_0^t sin(u)e^u du + y_0 e^{-t} = \frac{sin(t)}{2} - \frac{cos(t)}{2} + (y_0+0.5)e^{-t},$$

which agrees with the solution for integer case $y' + y = \sin(t),\ y(0) = y_0$. □

Theorem 3.6. *For* $0 < \alpha < 1, 1 < \beta < 2,$ *The solution of the differential equation*

$$\mathbb{D}_0^{\beta} y + \mathbb{D}_0^{\alpha} y + y = 0,\ y(0) = y_0, y'(0) = y_1,$$

is given by

$$\begin{aligned}
y(t) &= y_0\sum_{k=0}^{\infty} t^{\beta k}E_{\beta-\alpha,\beta k+1}^{k+1}(-t^{\beta-\alpha}) + y_1\sum_{k=0}^{\infty} t^{\beta k+1}E_{\beta-\alpha,\beta k+2}^{k+1}(-t^{\beta-\alpha}) \\
&+ y_0\sum_{k=0}^{\infty} t^{\beta k+\beta-\alpha}E_{\beta-\alpha,\beta k+\beta+1-\alpha}^{k+1}(-t^{\beta-\alpha})
\end{aligned}$$

Proof. The Laplace transform of the differential equation is

$$s^{\beta}F(s) - s^{\beta-1}y_0 - s^{\beta-2}y_1 + s^{\alpha}F(s) - s^{\alpha-1}y_0 + F(s) = 0.$$

Then

$$F(s) = \frac{s^{\beta-1}y_0}{(s^{\beta}+s^{\alpha})(1+\frac{1}{s^{\beta}+s^{\alpha}})} + \frac{s^{\beta-2}y_1}{(s^{\beta}+s^{\alpha})(1+\frac{1}{s^{\beta}+s^{\alpha}})} + \frac{s^{\alpha-1}y_0}{(s^{\beta}+s^{\alpha})(1+\frac{1}{s^{\beta}+s^{\alpha}})}.$$

It follows that

$$\begin{aligned}
F(s) &= y_0 s^{\beta-1}\sum_{k=0}^{\infty}(-1)^k(s^{\beta}+s^{\alpha})^{-(k+1)} + y_1 s^{\beta-2}\sum_{k=0}^{\infty}(-1)^k(s^{\beta}+s^{\alpha})^{-(k+1)} \\
&+ y_0 s^{\alpha-1}\sum_{k=0}^{\infty}(-1)^k(s^{\beta}+s^{\alpha})^{-(k+1)} \\
&= y_0\sum_{k=0}^{\infty}\frac{s^{\beta-1}(-1)^k}{(s^{\beta}+s^{\alpha})^{k+1}} + y_1\sum_{k=0}^{\infty}\frac{s^{\beta-2}(-1)^k}{(s^{\beta}+s^{\alpha})^{k+1}} + y_0\sum_{k=0}^{\infty}\frac{s^{\alpha-1}(-1)^k}{(s^{\beta}+s^{\alpha})^{k+1}} \\
&= y_0\sum_{k=0}^{\infty}s^{-(\beta k+1)}(1+s^{-(\beta-\alpha)})^{-(k+1)} + y_1\sum_{k=0}^{\infty}s^{-(\beta k+2)}(1+s^{-(\beta-\alpha)})^{-(k+1)} \\
&+ y_0\sum_{k=0}^{\infty}s^{-(\beta k+\beta+1-\alpha)}(1+s^{-(\beta-\alpha)})^{-(k+1)}
\end{aligned}$$

Hence the inverse Laplace transform is

$$\begin{aligned}
y(t) = \mathcal{L}^{-1}(F(s);t) &= y_0\sum_{k=0}^{\infty}t^{\beta k}E^{k+1}_{\beta-\alpha,\beta k+1}(-t^{\beta-\alpha}) + y_1\sum_{k=0}^{\infty}t^{\beta k+1}E^{k+1}_{\beta-\alpha,\beta k+2}(-t^{\beta-\alpha}) \\
&+ y_0\sum_{k=0}^{\infty}t^{\beta k+\beta-\alpha}E^{k+1}_{\beta-\alpha,\beta k+\beta+1-\alpha}(-t^{\beta-\alpha}).
\end{aligned}$$

□

Chapter 4

Application: Economic Growth Modeling

While Fractional Calculus has existed in the theoretical realm of mathematics as long as it's classical counterpart, the pragmatic applications of said branch of calculus were sparse until the last century, when a rather large number of scientific branches, such as physics, engineering and finance began applying fractional differential equations to problems in said fields. One such application is describing economic growth over large time periods, since fractional differential equations, more so than their integer counterparts, are suitable for establishing dynamic models for series where a memory effect may exist [4].

Tejado, D. Valerio, and N. Valerio apply a fractional order approach to measuring Spanish economic growth in their article *Fractional Calculus in Economic Growth Modeling. The Spanish Case.* In the article, they define the differential operator ${}_cD_t^n f(t) = \frac{d^n f(t)}{dt^n}$, and, by mathematical induction, define ${}_cD_t^n$ as

$${}_cD_t^n f(t) = \lim_{h\to\infty} \frac{\sum\limits_{k=0}^{n} (-1)^k \binom{n}{k} f(t-kh)}{h^n},$$

where $n \in \mathbb{N}$.

By definition of the Gamma function in $\mathbb{C} - \mathbb{Z}^-$, the authors generalize the differential operator for non-integer orders as

$${}_cD_t^\alpha f(t) = \lim_{h\to\infty} \frac{\sum\limits_{k=0}^{\lfloor \frac{t-c}{h} \rfloor} (-1)^k \binom{\alpha}{k} f(t-kh)}{h^\alpha},$$

where $\alpha \in \mathbb{R}$, and c and t are called terminals. Note that when $\alpha \in \mathbb{N}$, the fractional order equation will reduce to the stricly integer case when $h > 0$. Note also that $\lfloor \frac{t-c}{h} \rfloor$ was set as the upper limit so that when $\alpha \in \mathbb{Z}^-$, the fractional order approach becomes a Riemann integral [5].

The authors apply these definitions of integer and fractional order derivatives to a simple model of a national economy in the form

$$y(t) = f(x_1, x_2, ..., x_9)$$

where the endogenous variable y measures GDP in some given year t, while the exogenous variables x_k are the variables that the output depends on, which consist of:

- land area (x_1)
- arable land (x_2)
- total population (x_3)
- average years of school attendance (x_4)
- gross capital formation (GCF)(x_5)
- exports of goods and services (x_6)
- general government final consumption expenditure (GGFCE) (x_7)
- money and quasi money (x_8)
- investment ($x_9 \equiv x_5$)

Average years of school attendance was obtained from Fuente and Doménech's research in their article *Educational attainment in the OECD,1960–2010.* The rest of the variables were obtained from indicators in the World Data Bank. Thus, the authors of the main article considered the following integer and fracitonal order models:

$$y(t) = C_1 x_1(t) + C_2 x_2(t) + C_3 x_3(t) + C_4 x_4(t) + C_5 \int_{t_0}^{t} x_5(t)dt + C_6 x_6(t) + C_7 x_7(t) + C_8 \frac{dx_8(t)}{dt} + C_9 \frac{dx_9(t)}{dt}$$

$$y(t) = \sum_{k=1}^{9} C_k D^{\alpha_k} x_k(t)$$

4.0.1 Conclusion

For future research, we would like to incorporate both the integer and fractional derivative methods to the United States economic data in order to determine whether or not the fractional model better predicts economic growth in the United States. [4] and [5] concluded that, for both Portugal and Spain, respectively, the fractional model served as an overall better predictor for economic growth modeling than the integer case. Their fitting procedure was implemented in MATLAB, using *fminsearch*, which utilizes Nelder-Mead's simplex search method. They use this in order to minimize the mean square error (MSE), which then leads to finding that a fractional order influence is present on average years of schooling, gross capital formation, General government final consumption expenditure, quasi money, and investment. This leads to their final result that a model of fractional order better predicts economic growth modeling [4] [5].

By utilizing both cases, as well as their minimization methods such as *fminsearch*, to the United States Economic model, we would like to prove whether or not a fractional order model better predicts United States economic growth modeling, providing a robust result to a fractional order model predicting economic growth better than a traditional integer order model.

The following table consists of the United States economic data obtained from the World Data Bank from 1960 to 2013, as well as Fuente and Doménech data regarding average years of schooling [6]. GDP, x_5, x_6, x_7, and x_8 in current United States dollars, x_1 in km^2, x_2 in percentage of x_1, x_3 in people and x_4 in years. [7]

Table 4.1: United States Economic Data, 1960 - 2013

year	y	x1	x2	x3	x4	x5	x6	x7	x8
1960	5.43E+11	9158960	1814828.063	180671000	10.56	1.22E+11	2.70E+10	8.50E+10	3.26E+11
1961	5.63E+11	9158960	1806300	183691000	10.64216832	1.27E+11	2.76E+10	8.99E+10	3.53E+11
1962	6.05E+11	9158960	1770950	186538000	10.72182437	1.40E+11	2.91E+10	9.83E+10	3.85E+11
1963	6.39E+11	9158960	1795740	189242000	10.80137243	1.48E+11	3.11E+10	1.04E+11	4.20E+11
1964	6.86E+11	9158960	1779660	191889000	10.88075807	1.59E+11	3.50E+10	1.09E+11	4.58E+11
1965	7.44E+11	9158960	1770000	194303000	10.97	1.78E+11	3.71E+10	1.17E+11	4.98E+11
1966	8.15E+11	9158960	1757050	196560000	11.0388614	1.98E+11	4.09E+10	1.33E+11	5.21E+11
1967	8.62E+11	9158960	1744870	198712000	11.11769253	2.00E+11	4.35E+10	1.50E+11	5.75E+11
1968	9.43E+11	9158960	1810000	200706000	11.19658813	2.16E+11	4.79E+10	1.68E+11	6.25E+11
1969	1.02E+12	9158960	1892440	202677000	11.2757161	2.42E+11	5.19E+10	1.81E+11	6.31E+11
1970	1.08E+12	9158960	1887350	205052000	11.33	2.30E+11	5.97E+10	1.94E+11	7.02E+11
1971	1.17E+12	9158960	1881400	207661000	11.43524807	2.55E+11	6.30E+10	2.11E+11	8.00E+11
1972	1.28E+12	9158960	1875450	209896000	11.51543218	2.89E+11	7.08E+10	2.28E+11	9.09E+11
1973	1.43E+12	9158960	1870500	211909000	11.59540886	3.33E+11	9.53E+10	2.41E+11	1.00E+12
1974	1.55E+12	9158960	1864720	213854000	11.67479033	3.51E+11	1.27E+11	2.67E+11	1.08E+12
1975	1.69E+12	9158960	1864720	215973000	11.76	3.42E+11	1.39E+11	2.99E+11	1.19E+12
1976	1.88E+12	9158960	1864720	218035000	11.83024153	4.13E+11	1.50E+11	3.16E+11	1.31E+12
1977	2.09E+12	9158960	1865520	220239000	11.90568584	4.90E+11	1.59E+11	3.43E+11	1.47E+12
1978	2.36E+12	9158960	1887550	222585000	11.97928409	5.84E+11	1.87E+11	3.72E+11	1.63E+12
1979	2.63E+12	9158960	1887550	225055000	12.05079865	6.60E+11	2.30E+11	4.05E+11	1.79E+12
1980	2.86E+12	9158960	1887550	227225000	12.14	6.66E+11	2.81E+11	4.55E+11	1.99E+12
1981	3.21E+12	9158960	1887550	229466000	12.18669937	7.79E+11	3.05E+11	5.07E+11	2.23E+12
1982	3.34E+12	9158960	1877650	231664000	12.25104962	7.38E+11	2.83E+11	5.53E+11	2.45E+12
1983	3.64E+12	9158960	1877650	233792000	12.3132444	8.09E+11	2.77E+11	5.95E+11	2.65E+12
1984	4.04E+12	9158960	1877650	235825000	12.37348542	1.01E+12	3.02E+11	6.32E+11	2.98E+12
1985	4.35E+12	9158960	1877650	237924000	12.44	1.05E+12	3.03E+11	6.89E+11	3.23E+12
1986	4.59E+12	9158960	1877650	240133000	12.48894278	1.09E+12	3.21E+11	7.36E+11	3.53E+12
1987	4.87E+12	9158960	1857420	242289000	12.54473999	1.15E+12	3.64E+11	7.76E+11	3.68E+12
1988	5.25E+12	9158960	1857420	244499000	12.59974523	1.20E+12	4.45E+11	8.20E+11	3.93E+12
1989	5.66E+12	9158960	1857260	246819000	12.65433764	1.27E+12	5.04E+11	8.81E+11	4.14E+12
1990	5.98E+12	9158960	1856760	249623000	12.66	1.28E+12	5.52E+11	9.48E+11	4.25E+12
1991	6.17E+12	9158960	1856760	252981000	12.76362113	1.24E+12	5.95E+11	1.00E+12	4.31E+12
1992	6.54E+12	9158960	1840800	256514000	12.8179941	1.31E+12	6.33E+11	1.05E+12	4.30E+12
1993	6.88E+12	9158960	1827480	259919000	12.87131801	1.40E+12	6.55E+11	1.07E+12	4.33E+12
1994	7.31E+12	9158960	1819390	263126000	12.92289561	1.55E+12	7.21E+11	1.11E+12	4.35E+12
1995	7.66E+12	9158960	1818390	266278000	13.01	1.63E+12	8.13E+11	1.14E+12	4.65E+12
1996	8.10E+12	9158960	1790060	269394000	13.01816213	1.75E+12	8.68E+11	1.18E+12	5.01E+12
1997	8.61E+12	9158960	1775920	272657000	13.06129242	1.93E+12	9.54E+11	1.22E+12	5.41E+12
1998	9.09E+12	9158960	1767820	275854000	13.10155912	2.08E+12	9.53E+11	1.27E+12	5.93E+12
1999	9.66E+12	9158960	1753680	279040000	13.13910083	2.25E+12	9.92E+11	1.36E+12	6.50E+12
2000	1.03E+13	9161920	1753680	282162411	13.19	2.42E+12	1.10E+12	1.44E+12	7.02E+12
2001	1.06E+13	9161920	1754000	284968955	13.20662215	2.34E+12	1.03E+12	1.55E+12	7.55E+12
2002	1.10E+13	9161920	1729770	287625193	13.23722917	2.37E+12	1.00E+12	1.65E+12	7.88E+12
2003	1.15E+13	9161920	1716340	290107933	13.26636605	2.49E+12	1.04E+12	1.76E+12	8.23E+12
2004	1.23E+13	9161920	1670560	292805298	13.29452157	2.77E+12	1.18E+12	1.87E+12	8.70E+12
2005	1.31E+13	9161920	1651150	295516599	13.3	3.04E+12	1.31E+12	1.98E+12	9.41E+12
2006	1.39E+13	9161920	1604413	298379912	13.3497624	3.23E+12	1.48E+12	2.09E+12	1.03E+13
2007	1.45E+13	9161920	1618800	301231207	13.3773371	3.24E+12	1.66E+12	2.21E+12	1.15E+13
2008	1.47E+13	9147420	1630635	304093966	13.40490924	3.06E+12	1.84E+12	2.37E+12	1.24E+13
2009	1.44E+13	9147420	1605396	306771529	13.43247943	2.53E+12	1.59E+12	2.44E+12	1.30E+13
2010	1.50E+13	9147420	1598330	309347057	13.46	2.75E+12	1.85E+12	2.52E+12	1.27E+13
2011	1.55E+13	9147420	1601625	311721632	13.48761657	2.88E+12	2.11E+12	2.53E+12	1.35E+13
2012	1.62E+13	9147420	1551075	314112078	13.5151849	3.10E+12	2.19E+12	2.55E+12	1.42E+13
2013	1.68E+13	9147420	1526522.46	316497531	13.54275323	3.24E+12	2.26E+12	2.55E+12	1.48E+13

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