Homology of 3-dimensional Lie algebras

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1 Abstract

In this paper, we calculate the Lie algebra homology of all three dimensional Lie algebras in the classification provided by Bianchi [1].

2 Introduction

Recall from [2] that a Lie algebra \( g \) is a vector space over a field \( F \) together with a binary operation 
\[
[\cdot, \cdot] : g \times g \longrightarrow g
\]
called the Lie bracket, which satisfies the following axioms:

a) \([ax + by, z] = a [x, y] + b [y, z]\), \([z, ax + by] = a [z, x] + b [z, y]\) for all scalars \( a, b \) in \( F \) and all elements \( x, y, z \) in \( g \).

b) \([x, y] = -[y, x]\) for all elements \( x, y \) in \( g \).

c) \([x, [y, z]] + y, [z, x] + [z, [x, y]] = 0\) for all \( x, y, z \) in \( g \), when Characteristic \( F \) is not 2.

Recall also that for a Lie algebra \( g \), the Lie algebra homology of \( g \) with coefficients in \( \mathbb{R} \), written \( H_{*}^{Lie}(g; \mathbb{R}) \) is the homology of the Chevalley-Eilenberg complex \( \wedge^{*}(g; \mathbb{R}) \), namely
\[
\mathbb{R} \leftarrow \partial \leftarrow g \leftarrow \wedge^{2} g \leftarrow \cdots \leftarrow \wedge^{n-1} g \leftarrow \wedge^{n} g \leftarrow \partial \leftarrow \cdots
\]
where \( g^{\wedge n} \) is the \( n \)th exterior power of \( g \) over \( k \), and where
\[
\partial(g_{1} \wedge \cdots \wedge g_{n}) = \sum_{1 \leq i < j \leq n} (-1)^{i+j-1} [g_{i}, g_{j}] \wedge g_{1} \wedge \cdots \hat{g}_{i} \cdots \hat{g}_{j} \cdots \wedge g_{n}
\]
where \( \hat{g}_{i} \) means that the variable \( g_{i} \) is deleted.

\[
H_{*}^{Lie}(g; \mathbb{R}) = \frac{\ker(\partial_{*})}{\text{Im}(\partial_{*+1})}.
\]

For simplicity, in this paper we will denote \( H_{*}^{Lie}(g; \mathbb{R}) \) by \( H_{*}^{Lie}(g) \).
3 Homology of 3-dimensional Lie algebras

In 1898 [1], Bianchi provided a classification of three-dimensional Lie algebras and proved that any three-dimensional Lie algebra has the same structure as a Lie algebra in this classification. The following is Bianchi's classification:

Theorem 3.1 (Bianchi): Let \( g \) be a real 3-dimensional Lie algebra. Then \( g \) is isomorphic to one of the following Lie algebras:

- **Bianchi I:** \([e_1, e_2] = [e_2, e_3] = [e_3, e_1] = 0\)
- **Bianchi II:** \([e_1, e_2] = 0, [e_2, e_3] = e_1, [e_3, e_1] = 0\)
- **Bianchi IV:** \([e_1, e_2] = 0, [e_2, e_3] = e_1 - e_2, [e_3, e_1] = e_1\)
- **Bianchi V:** \([e_1, e_2] = 0, [e_2, e_3] = e_2, [e_3, e_1] = e_1\)
- **Bianchi VI\(_h\)(\( h \leq 0 \):) \([e_1, e_2] = 0, [e_2, e_3] = e_1 - he_2, [e_3, e_1] = he_1 - e_2\)
- **Bianchi VII\(_h\)(\( h \geq 0 \):) \([e_1, e_2] = 0, [e_2, e_3] = e_1 - he_2, [e_3, e_1] = he_1 + e_2\)
- **Bianchi VIII:** \([e_1, e_2] = -e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2\)
- **Bianchi IX:** \([e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2\)

In the next theorems, we provide explicit calculations of the Lie algebra homology of the Lie algebras in this classification.

Theorem 3.2 (Bianchi I): Let \( g = \text{span}\{e_1, e_2, e_3\} \) be a Lie algebra isomorphic to Bianchi II, given by the brackets \([e_1, e_2] = 0, [e_2, e_3] = 0, [e_3, e_1] = 0\). Then,

\[
H^\text{Lie}_*(g) = \wedge^* g
\]

Proof. The Chevalley-Eilenberg complex is reduced to

\[
0 \xleftarrow{\partial_0} \mathbb{R} \xleftarrow{\partial_1} g \xleftarrow{\partial_2} g^{\wedge_2} \xleftarrow{\partial_3} g^{\wedge_3} \leftarrow 0
\]

\[
g = \text{span}\{e_1, e_2, e_3\}
\]

\[
g^{\wedge_2} = \text{span}\{e_1 \wedge e_2, e_2 \wedge e_3, e_3 \wedge e_1\}
\]

\[
g^{\wedge_3} = \text{span}\{e_1 \wedge e_2 \wedge e_3\}
\]

\[
g^{\wedge_4} = 0
\]

\[
\partial_1(e_1) = 0, \partial_1(e_2) = 0, \partial_1(e_3) = 0
\]

\[
\partial_2(e_1 \wedge e_2) = [e_1, e_2] = 0
\]

\[
\partial_2(e_2 \wedge e_3) = [e_2, e_3] = 0
\]

\[
\partial_2(e_1 \wedge e_3) = [e_1, e_3] = 0
\]

\[
\partial_3(e_1 \wedge e_2 \wedge e_3) = (-1)^4[e_1, e_2] \wedge e_3 + (-1)^5[e_1, e_3] \wedge e_2 + (-1)^6[e_2, e_3] \wedge e_1
\]

\[
= 0
\]

\[
H_k^\text{Lie}(g) = \frac{\ker \partial_k}{\im \partial_{k+1}} = g^{\wedge_k}/g = g^{\wedge_k}.
\]

\(\square\)
\textbf{Theorem 3.3 (Bianchi II):} Let $\mathfrak{g} = \text{span}\{e_1, e_2, e_3\}$ be a Lie algebra isomorphic to Bianchi II, given by the brackets $[e_1, e_2] = 0, [e_2, e_3] = e_1, [e_3, e_1] = 0$. Then,

$$H^\text{Lie}_k(\mathfrak{g}) = \begin{cases} \mathbb{R}, & \text{if } k = 0 \\ \langle e_2, e_3 \rangle, & \text{if } k = 1 \\ \langle e_1 \wedge e_2, e_1 \wedge e_3 \rangle, & \text{if } k = 2 \\ \langle e_1 \wedge e_2 \wedge e_3 \rangle, & \text{if } k = 3 \\ 0, & \text{if } k > 3 \end{cases}$$

\textbf{Proof.} The Chevalley-Eilenberg complex is reduced to

\[ 0 \xleftarrow{\partial_0} \mathbb{R} \xleftarrow{\partial_1} \mathfrak{g} \xleftarrow{\partial_2} \mathfrak{g}^\wedge 2 \xleftarrow{\partial_2} \mathfrak{g}^\wedge 3 \xleftarrow{\partial_3} \cdots \]

$\mathfrak{g} = \text{span}\{e_1, e_2, e_3\}$

$\mathfrak{g}^\wedge 2 = \text{span}\{e_1 \wedge e_2, e_2 \wedge e_3, e_1 \wedge e_3\}$

$\mathfrak{g}^\wedge 3 = \text{span}\{e_1 \wedge e_2 \wedge e_3\}$

$\mathfrak{g}^\wedge 4 = 0$

$\partial_1(e_1) = 0, \partial_1(e_2) = 0, \partial_1(e_3) = 0$

$\partial_2(e_1 \wedge e_2) = [e_1, e_2] = 0$

$\partial_2(e_2 \wedge e_3) = [e_2, e_3] = e_1$

$\partial_2(e_1 \wedge e_3) = [e_1, e_3] = 0$

$\partial_3(e_1 \wedge e_2 \wedge e_3) = (-1)^4[e_1, e_2] \wedge e_3 + (-1)^5[e_1, e_3] \wedge e_2 + (-1)^6[e_2, e_3] \wedge e_1$

$= e_1 \wedge e_1$

$= 0$

$H^\text{Lie}_0(\mathfrak{g}) = \ker \partial_0 / \text{im} \partial_0 = \mathbb{R} / \{0\} = \mathbb{R}$

$H^\text{Lie}_1(\mathfrak{g}) = \ker \partial_1 / \text{im} \partial_1 = \langle e_1, e_2, e_3 \rangle / \langle e_1 \rangle = \langle e_2, e_3 \rangle$

$H^\text{Lie}_2(\mathfrak{g}) = \ker \partial_2 / \text{im} \partial_2 = \langle e_1 \wedge e_2, e_1 \wedge e_3 \rangle / \langle e_1 \rangle = \langle e_1 \wedge e_2, e_1 \wedge e_3 \rangle$

$H^\text{Lie}_3(\mathfrak{g}) = \ker \partial_3 / \text{im} \partial_3 = \langle e_1 \wedge e_2 \wedge e_3 \rangle / \{0\} = \langle e_1 \wedge e_2 \wedge e_3 \rangle$

\[ \square \]

\textbf{Theorem 3.4 (Bianchi IV):} Let $\mathfrak{g} = \text{span}\{e_1, e_2, e_3\}$ be a Lie algebra isomorphic to Bianchi IV, given by the brackets $[e_1, e_2] = 0, [e_2, e_3] = e_1 - e_2, [e_3, e_1] = e_1$. Then,

$$H^\text{Lie}_k(\mathfrak{g}) = \begin{cases} \mathbb{R}, & \text{if } k = 0 \\ \langle e_3 \rangle, & \text{if } k = 1 \\ 0, & \text{if } k > 1 \end{cases}$$

\textbf{Proof.} The Chevalley-Eilenberg complex is reduced to

\[ 0 \xleftarrow{\partial_0} \mathbb{R} \xleftarrow{\partial_1} \mathfrak{g} \xleftarrow{\partial_2} \mathfrak{g}^\wedge 2 \xleftarrow{\partial_3} \mathfrak{g}^\wedge 3 \xleftarrow{\partial_4} \cdots \]
\( \mathfrak{g} = \{ e_1, e_2, e_3 \} \)
\( \mathfrak{g}^{\wedge 2} = \{ e_1 \wedge e_2, e_2 \wedge e_3, e_1 \wedge e_3 \} \)
\( \mathfrak{g}^{\wedge 3} = \{ e_1 \wedge e_2 \wedge e_3 \} \)
\( \mathfrak{g}^{\wedge 4} = 0 \)

\( \partial_1(e_1) = 0, \partial_1(e_2) = 0, \partial_1(e_3) = 0 \)
\( \partial_2(e_1 \wedge e_2) = [e_1, e_2] = 0 \)
\( \partial_2(e_2 \wedge e_3) = [e_2, e_3] = e_1 - e_2 \)
\( \partial_2(e_1 \wedge e_3) = [e_1, e_3] = -e_1 \)

\( \partial_3(e_1 \wedge e_2 \wedge e_3) = (-1)^4[e_1, e_2] \wedge e_3 + (-1)^6[e_2, e_3] \wedge e_1 + (-1)^5[e_1, e_3] \wedge e_2 \)
\( = (e_1 - e_2) \wedge e_1 + e_1 \wedge e_2 \)
\( = e_1 \wedge e_1 - e_2 \wedge e_1 + e_1 \wedge e_2 \)
\( = 2(e_1 \wedge e_2) \)

\[
H_0^{\text{Lie}}(\mathfrak{g}) = \ker \partial_0 \overset{\text{im} \partial_0}{\longrightarrow} \mathbb{R} \cong \mathbb{R}
\]
\[
H_1^{\text{Lie}}(\mathfrak{g}) = \ker \partial_1 \overset{\text{im} \partial_1}{\longrightarrow} \langle e_1, e_2, e_3 \rangle = \langle e_3 \rangle
\]
\[
H_2^{\text{Lie}}(\mathfrak{g}) = \ker \partial_2 \overset{\text{im} \partial_2}{\longrightarrow} \langle e_1 \wedge e_2 \rangle = \langle 0 \rangle
\]
\[
H_3^{\text{Lie}}(\mathfrak{g}) = \ker \partial_3 \overset{\text{im} \partial_3}{\longrightarrow} \langle e_1 \wedge e_2 \wedge e_3 \rangle = \langle 0 \rangle
\]

\[\square\]

**Theorem 3.5 (Bianchi V):** Let \( \mathfrak{g} = \text{span}\{e_1, e_2, e_3\} \) be a Lie algebra isomorphic to Bianchi V, given by the brackets \([e_1, e_2] = 0, [e_2, e_3] = e_2, [e_3, e_1] = e_1\). Then,

\[
H_k^{\text{Lie}}(\mathfrak{g}) = \begin{cases} 
\mathbb{R}, & \text{if } k=0 \\
\langle e_3 \rangle, & \text{if } k=1 \\
\langle e_1 \wedge e_2 \rangle, & \text{if } k=2 \\
\langle e_1 \wedge e_2 \wedge e_3 \rangle, & \text{if } k=3 \\
0, & \text{if } k > 3 
\end{cases}
\]

**Proof.** The Chevalley-Eilenberg complex is reduced to

\[
0 \overset{\partial_0}{\leftarrow} \mathbb{R} \overset{\partial_1}{\leftarrow} \mathfrak{g} \overset{\partial_2}{\leftarrow} \mathfrak{g}^{\wedge 2} \overset{\partial_3}{\leftarrow} \mathfrak{g}^{\wedge 3} \leftarrow \cdots
\]

\( \mathfrak{g} = \{ e_1, e_2, e_3 \} \)
\( \mathfrak{g}^{\wedge 2} = \{ e_1 \wedge e_2, e_2 \wedge e_3, e_1 \wedge e_3 \} \)
\( \mathfrak{g}^{\wedge 3} = \{ e_1 \wedge e_2 \wedge e_3 \} \)
\( \mathfrak{g}^{\wedge 4} = 0 \)

\( \partial_1(e_1) = 0, \partial_1(e_2) = 0, \partial_1(e_3) = 0 \)
\( \partial_2(e_1 \wedge e_2) = [e_1, e_2] = 0 \)
\( \partial_2(e_2 \wedge e_3) = [e_2, e_3] = e_2 \)
\( \partial_2(e_1 \wedge e_3) = [e_1, e_3] = -e_1 \)
\[\partial_2(e_1 \wedge e_3) = [e_1, e_3] = -e_1\]

\[\partial_3(e_1 \wedge e_2 \wedge e_3) = (-1)^4[e_1, e_2] \wedge e_3 + (-1)^6[e_2, e_3] \wedge e_1 + (-1)^5[e_1, e_3] \wedge e_2\]

\[= e_2 \wedge e_1 + e_1 \wedge e_2\]

\[= 0\]

\[H^\text{Lie}_0(g) = \ker \partial_0 \overset{\dim g}{=} \mathbb{R}\]

\[H^\text{Lie}_1(g) = \ker \partial_1 \overset{\dim g}{=} \langle e_1 + e_2, e_3 \rangle = \langle e_3 \rangle\]

\[H^\text{Lie}_2(g) = \ker \partial_2 \overset{\dim g}{=} \langle e_1 \wedge e_2 \rangle = \langle e_1 \wedge e_2 \rangle\]

\[H^\text{Lie}_3(g) = \ker \partial_3 \overset{\dim g}{=} \langle e_1 \wedge e_2 \wedge e_3 \rangle = \langle e_1 \wedge e_2 \wedge e_3 \rangle\]

\[\square\]

**Theorem 3.6 (Bianchi VI):** Let \( g = \text{span}\{e_1, e_2, e_3\} \) be a Lie algebra isomorphic to Bianchi VI, given by the brackets \( h \leq 0, [e_1, e_2] = 0, [e_2, e_3] = e_1 - he_2, [e_3, e_1] = he_1 - e_2 \). Then,

\[H^\text{Lie}_k(g) = \begin{cases} \mathbb{R}, & \text{if } k=0 \\ \langle e_1 + e_2, e_3 \rangle, & \text{if } k=1 \text{ and } h = -1 \\ \langle e_3 \rangle, & \text{if } k=1 \text{ and } h \neq -1 \\ \langle e_1 \wedge e_3 - e_2 \wedge e_3 \rangle, & \text{if } k=2 \text{ and } h = -1 \\ \langle e_1 \wedge e_2 \rangle, & \text{if } k=2 \text{ and } h = 0 \\ \langle 0 \rangle, & \text{if } k=2 \text{ and } h \neq -1 \text{ and } h \neq 0 \\ \langle e_1 \wedge e_2 \wedge e_3 \rangle, & \text{if } k=3 \text{ and } h = 0 \\ 0, & \text{else.} \end{cases}\]

**Proof.** The Chevalley-Eilenberg complex is reduced to

\[0 \leftarrow \mathbb{R} \leftarrow g \leftarrow g^2 \leftarrow g^3 \leftarrow \cdots\]

\[g = \{e_1, e_2, e_3\}\]

\[g^2 = \{e_1 \wedge e_2, e_2 \wedge e_3, e_1 \wedge e_3\}\]

\[g^3 = \{e_1 \wedge e_2 \wedge e_3\}\]

\[g^4 = 0\]

\[\partial_1(e_1) = 0, \partial_1(e_2) = 0, \partial_1(e_3) = 0\]

\[\partial_2(e_1 \wedge e_2) = [e_1, e_2] = 0\]

\[\partial_2(e_2 \wedge e_3) = [e_2, e_3] = e_1 - he_2\]

\[\partial_2(e_1 \wedge e_3) = [e_1, e_3] = -he_1 + e_2\]

\[\partial_3(e_1 \wedge e_2 \wedge e_3) = (-1)^4[e_1, e_2] \wedge e_3 + (-1)^6[e_2, e_3] \wedge e_1 + (-1)^5[e_1, e_3] \wedge e_2\]

\[= (e_1 - he_2) \wedge e_1 + (he_1 - e_2) \wedge e_2\]

\[= e_1 \wedge e_1 - he_2 \wedge e_1 + he_1 \wedge e_2 - e_2 \wedge e_2\]

\[= 2h(e_1 \wedge e_2)\]
Note that $\ker \partial_2 = \langle e_1 \wedge e_2, e_1 \wedge e_3 - e_2 \wedge e_3 \rangle$ if $h = -1$, and $\ker \partial_2 = \langle e_1 \wedge e_2 \rangle$ if $h = 0$. Also, $\ker \partial_3 = \langle e_1 \wedge e_2 \wedge e_3 \rangle$ if $h = 0$, and $\ker \partial_2 = \langle 0 \rangle$ if $h \neq 0$. We then have

$$H_0^{\text{Lie}}(\mathfrak{g}) = \frac{\ker \partial_2}{\text{im} \partial_1} = \mathbb{R} \langle 0 \rangle = \mathbb{R}$$

$$H_1^{\text{Lie}}(\mathfrak{g}) = \frac{\ker \partial_1}{\text{im} \partial_2} = \frac{\langle e_1, e_2, e_3 \rangle}{\langle e_1 - h e_2, -h e_1 + e_2 \rangle} = \begin{cases} \langle e_1, e_2, e_3 \rangle, & \text{if } h = -1 \\ \langle e_3 \rangle, & \text{if } h \neq -1, \end{cases}$$

$$H_2^{\text{Lie}}(\mathfrak{g}) = \frac{\ker \partial_2}{\text{im} \partial_3} = \frac{\ker \partial_1}{(2h(e_1 \wedge e_2))} = \begin{cases} \langle e_1 \wedge e_3 - e_2 \wedge e_3 \rangle, & \text{if } h = -1 \\ \langle e_1 \wedge e_2 \rangle, & \text{if } h = 0 \\ \langle 0 \rangle, & \text{if } h \neq -1, \end{cases}$$

$$H_3^{\text{Lie}}(\mathfrak{g}) = \frac{\ker \partial_3}{\text{im} \partial_4} = \begin{cases} \langle e_1 \wedge e_2 \wedge e_3 \rangle, & \text{if } h = 0 \\ 0, & \text{if } h \neq 0. \end{cases}$$

**Theorem 3.7 (Bianchi VII\(_h\)):** Let $\mathfrak{g} = \text{span} \{e_1, e_2, e_3\}$ be a Lie algebra isomorphic to Bianchi VII\(_h\), given by the brackets $h \geq 0$, $[e_1, e_2] = 0$, $[e_2, e_3] = e_1 - h e_2$, $[e_3, e_1] = h e_1 + e_2$. Then,

$$H_k^{\text{Lie}}(\mathfrak{g}) = \begin{cases} \mathbb{R}, & \text{if } k=0 \\ \langle e_3 \rangle, & \text{if } k=1 \\ \langle e_1 \wedge e_2 \rangle, & \text{if } k=2 \text{ and } h = 0 \\ 0, & \text{if } k=2 \text{ and } h \neq 0 \\ \langle e_1 \wedge e_2 \wedge e_3 \rangle, & \text{if } k=3 \text{ and } h = 0 \\ 0, & \text{else}. \end{cases}$$

**Proof.** The Chevalley-Eilenberg complex is reduced to

$$0 \xleftarrow{\partial_0} \mathbb{R} \xleftarrow{\partial_1} \mathfrak{g} \xleftarrow{\partial_2} \mathfrak{g}^\wedge 2 \xleftarrow{\partial_3} \mathfrak{g}^\wedge 3 \xleftarrow{\partial_4} \cdots$$

$\mathfrak{g} = \{e_1, e_2, e_3\}$

$\mathfrak{g}^\wedge 2 = \{e_1 \wedge e_2, e_2 \wedge e_3, e_1 \wedge e_3\}$

$\mathfrak{g}^\wedge 3 = \{e_1 \wedge e_2 \wedge e_3\}$

$\mathfrak{g}^\wedge 4 = 0$

$\partial_1(e_1) = 0, \partial_1(e_2) = 0, \partial_1(e_3) = 0$

$\partial_2(e_1 \wedge e_2) = [e_1, e_2] = 0$

$\partial_2(e_2 \wedge e_3) = [e_2, e_3] = e_1 - h e_2$

$\partial_2(e_1 \wedge e_3) = [e_1, e_3] = -h e_1 - e_2$

$\partial_3(e_1 \wedge e_2 \wedge e_3) = (-1)^4 [e_1, e_2] \wedge e_3 + (-1)^6 [e_2, e_3] \wedge e_1 + (-1)^5 [e_1, e_3] \wedge e_2$

$$= (e_1 - h e_2) \wedge e_1 + (h e_1 + e_2) \wedge e_2$$

$$= e_1 \wedge e_1 - h e_2 \wedge e_1 + h e_1 \wedge e_2 + e_2 \wedge e_2$$

$$= 2h(e_1 \wedge e_2)$$
Note that $\ker \partial_3 = \langle e_1 \wedge e_2 \wedge e_3 \rangle$ if $h = 0$, and $\ker \partial_3 = \langle 0 \rangle$ if $h \neq 0$. We then have

\[
H_0^{\text{Lie}}(g) = \ker \partial_0 = \frac{\mathbb{R}}{\langle 0 \rangle} = \mathbb{R}
\]

\[
H_1^{\text{Lie}}(g) = \ker \partial_1 = \frac{\langle e_1, e_2, e_3 \rangle}{\langle -e_1, e_2, -e_3 \rangle} = \langle e_3 \rangle
\]

\[
H_2^{\text{Lie}}(g) = \ker \partial_2 = \frac{\langle e_1 \wedge e_2 \rangle}{\langle 2h(e_1 \wedge e_2) \rangle} = \begin{cases} \langle e_1 \wedge e_2 \rangle, & \text{if } h = 0 \\ 0, & \text{if } h \neq 0 \end{cases}
\]

\[
H_3^{\text{Lie}}(g) = \ker \partial_3 = \frac{\ker \partial_3}{\langle 0 \rangle} = \begin{cases} \langle e_1 \wedge e_2 \wedge e_3 \rangle, & \text{if } h = 0 \\ 0, & \text{else} \end{cases}
\]

\[\square\]

**Theorem 3.8 (Bianchi VIII):** Let $g = \text{span}\{e_1, e_2, e_3\}$ be a Lie algebra isomorphic to Bianchi VIII, given by the brackets $[e_1, e_2] = -e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$. Then,

\[
H_k^{\text{Lie}}(g) = \begin{cases} \mathbb{R}, & \text{if } k = 0 \\ \langle e_1 \wedge e_2 \wedge e_3 \rangle, & \text{if } k = 3 \\ 0, & \text{if } k > 0 \text{ and } k \neq 3 \end{cases}
\]

**Proof.** The Chevalley-Eilenberg complex is reduced to

\[
0 \xleftarrow{\partial_0} \mathbb{R} \xleftarrow{\partial_1} g \xleftarrow{\partial_2} g^2 \xleftarrow{\partial_3} g^3 \xleftarrow{\partial_4} \cdots
\]

\[
g = \{e_1, e_2, e_3\}
\]

\[
g^2 = \{e_1 \wedge e_2, e_2 \wedge e_3, e_1 \wedge e_3\}
\]

\[
g^3 = \{e_1 \wedge e_2 \wedge e_3\}
\]

\[
g^4 = 0
\]

\[
\partial_1(e_1) = 0, \quad \partial_1(e_2) = 0, \quad \partial_1(e_3) = 0
\]

\[
\partial_2(e_1 \wedge e_2) = [e_1, e_2] = -e_3
\]

\[
\partial_2(e_2 \wedge e_3) = [e_2, e_3] = e_1
\]

\[
\partial_2(e_1 \wedge e_3) = [e_1, e_3] = -e_2
\]

\[
\partial_3(e_1 \wedge e_2 \wedge e_3) = (-1)^4[e_1, e_2] \wedge e_3 + (-1)^6[e_2, e_3] \wedge e_1 + (-1)^5[e_1, e_3] \wedge e_2
\]

\[
= -e_3 \wedge e_3 + e_1 \wedge e_1 + e_2 \wedge e_2
\]

\[
= 0
\]

\[
H_0^{\text{Lie}}(g) = \ker \partial_0 = \frac{\mathbb{R}}{\langle 0 \rangle} = \mathbb{R}
\]

\[
H_1^{\text{Lie}}(g) = \ker \partial_1 = \frac{\langle e_1, e_2, e_3 \rangle}{\langle -e_1, e_2, -e_3 \rangle} = \langle 0 \rangle
\]

\[
H_2^{\text{Lie}}(g) = \ker \partial_2 = \frac{\langle e_1 \wedge e_2 \rangle}{\langle 0 \rangle} = \langle 0 \rangle
\]

\[
H_3^{\text{Lie}}(g) = \ker \partial_3 = \frac{\langle e_1 \wedge e_2 \wedge e_3 \rangle}{\langle 0 \rangle} = \langle e_1 \wedge e_2 \wedge e_3 \rangle
\]

\[\square\]
Theorem 3.9 (Bianchi IX): Let $\mathfrak{g} = \text{span}\{e_1, e_2, e_3\}$ be a Lie algebra isomorphic to Bianchi IX, given by the brackets $[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$. Then,

$$H^\text{Lie}_k(\mathfrak{g}) = \begin{cases} \mathbb{R}, & \text{if } k = 0 \\ \langle e_1 \wedge e_2 \wedge e_3 \rangle, & \text{if } k = 3 \\ 0, & \text{if } k > 0 \text{ and } k \neq 3 \end{cases}$$

Proof. The Chevalley-Eilenberg complex is reduced to

$$0 \leftarrow \mathbb{R} \leftarrow \mathfrak{g} \leftarrow \mathfrak{g}^\wedge 2 \leftarrow \mathfrak{g}^\wedge 3 \leftarrow \mathfrak{g}^\wedge 4 \leftarrow \cdots$$

$\mathfrak{g} = \{e_1, e_2, e_3\}$
$\mathfrak{g}^\wedge 2 = \{e_1 \wedge e_2, e_2 \wedge e_3, e_1 \wedge e_3\}$
$\mathfrak{g}^\wedge 3 = \{e_1 \wedge e_2 \wedge e_3\}$
$\mathfrak{g}^\wedge 4 = 0$

$\partial_1(e_1) = 0, \partial_1(e_2) = 0, \partial_1(e_3) = 0$
$\partial_2(e_1 \wedge e_2) = \langle e_1 \wedge e_2 \rangle = [e_1, e_2] = e_3$
$\partial_2(e_2 \wedge e_3) = \langle e_2 \wedge e_3 \rangle = [e_2, e_3] = e_1$
$\partial_2(e_1 \wedge e_3) = \langle e_1 \wedge e_3 \rangle = [e_1, e_3] = -e_2$

$$\partial_3(e_1 \wedge e_2 \wedge e_3) = (-1)^4[e_1, e_2] \wedge e_3 + (-1)^6[e_2, e_3] \wedge e_1 + (-1)^5[e_1, e_3] \wedge e_2$$
$$= e_3 \wedge e_1 + e_1 \wedge e_1 + e_2 \wedge e_2$$
$$= 0$$

$$H^\text{Lie}_0(\mathfrak{g}) = \ker \partial_0 / \text{im} \partial_0 = \mathbb{R} / \langle 0 \rangle = \mathbb{R}$$
$$H^\text{Lie}_1(\mathfrak{g}) = \ker \partial_1 / \text{im} \partial_2 = \langle (e_1, e_2, e_3) \rangle / \langle (e_3, e_1, -e_2) \rangle = \langle 0 \rangle$$
$$H^\text{Lie}_2(\mathfrak{g}) = \ker \partial_2 / \text{im} \partial_3 = \langle (0) \rangle / \langle 0 \rangle = \langle 0 \rangle$$
$$H^\text{Lie}_3(\mathfrak{g}) = \ker \partial_3 / \text{im} \partial_4 = \langle (e_1 \wedge e_2 \wedge e_3) \rangle / \langle (e_1 \wedge e_2 \wedge e_3) \rangle = \langle e_1 \wedge e_2 \wedge e_3 \rangle$$

Remark 3.10 From the theorems above, one notices that the Lie algebras Bianchi V, Bianchi VI$_h$ and Bianchi VII$_h$ have the same homology when $h = 0$, Bianchi IV, and Bianchi VII$_h$ have the same homology when $h \neq 0$, and the Lie algebras Bianchi VIII and Bianchi IX have the same homology.

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References
