Homology of 3-dimensional Lie algebras

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1 Abstract

In this paper, we calculate the Lie algebra homology of all three dimensional Lie algebras in the classification provided by **Bianchi** [1].

2 Introduction

Recall from [2] that a Lie algebra $\mathfrak{g}$ is a vector space over a field F together with a binary operation

$$[.,.] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

called the Lie bracket, which satisfies the following axioms:

a) $[ax + by, z] = a\,[x, y] + b\,[y, z]\,, \quad [z, ax + by] = a\,[z, x] + b\,[z, y]$ for all scalars a, b in F and all elements x, y, z in $\mathfrak{g}$.

b) $[x, y] = -\,[y, x]$ for all elements x, y in $\mathfrak{g}$.

c) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all x, y, z in $\mathfrak{g}$, when Characteristic F is not 2.

Recall also that for a Lie algebra $\mathfrak{g}$, the Lie algebra homology of $\mathfrak{g}$ with coefficients in $\mathbb{R}$, written $H_*^{Lie}(\mathfrak{g}; \mathbb{R})$ is the homology of the Chevalley-Eilenberg complex $\wedge^*(\mathfrak{g}; \mathbb{R})$, namely

$$\mathbb{R} \overset{\partial}{\leftarrow} \mathfrak{g} \overset{\partial}{\leftarrow} \mathfrak{g}^{\wedge^2} \overset{\partial}{\leftarrow} \cdots \overset{\partial}{\leftarrow} \mathfrak{g}^{\wedge^{n-1}} \overset{\partial}{\leftarrow} \mathfrak{g}^{\wedge^n} \overset{\partial}{\leftarrow} \cdots$$

where $\mathfrak{g}^{\wedge^n}$ is the nth exterior power of g over k, and where

$$\partial(g_1 \wedge \cdots \wedge g_n) = \sum_{1 \le i < j \le n} (-1)^{i+j-1} [g_i, g_j] \wedge g_1 \wedge \cdots \hat{g}_i \cdots \hat{g}_j \cdots \wedge g_n$$

where $\hat{g}_i$ means that the variable g_i is deleted.

$$H_*^{Lie}(\mathfrak{g}; \mathbb{R}) = \frac{ker(\partial_*)}{Im(\partial_{*+1})}. \tag{1}$$

For simplicity, in this paper we will denote $H_*^{Lie}(\mathfrak{g}; \mathbb{R})$ by $H_*^{Lie}(\mathfrak{g})$.

3 Homology of 3-dimensional Lie algebras

In 1898 [1], **Bianchi** provided a classification of three dimensional Lie algebras and proved that any three dimensional Lie algebra has the same structure as a Lie algebra in this classification. The following is **Bianchi**'s classification:

Theorem 3.1 *(**Bianchi**): Let $\mathfrak{g}$ be a real 3-dimensional Lie algebra. Then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras:*

***Bianchi** I:* $[e_1, e_2] = [e_2, e_3] = [e_3, e_1] = 0$
***Bianchi** II:* $[e_1, e_2] = 0, [e_2, e_3] = e_1, [e_3, e_1] = 0$
***Bianchi** IV:* $[e_1, e_2] = 0, [e_2, e_3] = e_1 - e_2, [e_3, e_1] = e_1$
***Bianchi** V:* $[e_1, e_2] = 0, [e_2, e_3] = e_2, [e_3, e_1] = e_1$
***Bianchi** $VI_h(h \leq 0)$:* $[e_1, e_2] = 0, [e_2, e_3] = e_1 - he_2, [e_3, e_1] = he_1 - e_2$
***Bianchi** $VII_h(h \geq 0)$:* $[e_1, e_2] = 0, [e_2, e_3] = e_1 - he_2, [e_3, e_1] = he_1 + e_2$
***Bianchi** VIII:* $[e_1, e_2] = -e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$
***Bianchi** IX:* $[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$

In the next theorems, we provide explicit calculations of the lie algebra homology of the lie algebras in this classification.

Theorem 3.2 *(**Bianchi** I): Let $\mathfrak{g} = span\{e_1, e_2, e_3\}$ be a Lie algebra isomorphic to **Bianchi** II, given by the brackets $[e_1, e_2] = 0, [e_2, e_3] = 0, [e_3, e_1] = 0$. Then,*

$$H_*^{Lie}(\mathfrak{g}) = \wedge^* \mathfrak{g}$$

Proof. The Chevalley-Eilenberg complex is reduced to

$0 \overset{\partial_0}{\longleftarrow} \mathbb{R} \overset{\partial_1}{\longleftarrow} \mathfrak{g} \overset{\partial_2}{\longleftarrow} \mathfrak{g}^{\wedge^2} \overset{\partial_3}{\longleftarrow} \mathfrak{g}^{\wedge^3} \leftarrow 0$

$\mathfrak{g} = span\{e_1, e_2, e_3\}$
$\mathfrak{g}^{\wedge 2} = span\{e_1 \wedge e_2, e_2 \wedge e_3, e_1 \wedge e_3\}$
$\mathfrak{g}^{\wedge 3} = span\{e_1 \wedge e_2 \wedge e_3\}$
$\mathfrak{g}^{\wedge 4} = 0$

$\partial_1(e_1) = 0,\ \partial_1(e_2) = 0,\ \partial_1(e_3) = 0$
$\partial_2(e_1 \wedge e_2) = [e_1, e_2] = 0$
$\partial_2(e_2 \wedge e_3) = [e_2, e_3] = 0$
$\partial_2(e_1 \wedge e_3) = [e_1, e_3] = 0$

$$\begin{aligned}\partial_3(e_1 \wedge e_2 \wedge e_3) &= (-1)^4[e_1, e_2] \wedge e_3 + (-1)^5[e_1, e_3] \wedge e_2 + (-1)^6[e_2, e_3] \wedge e_1 \\ &= 0\end{aligned}$$

$H_k^{Lie}(\mathfrak{g}) = \frac{ker\partial_k}{im\partial_{k+1}} = \frac{\mathfrak{g}^{\wedge k}}{\langle 0 \rangle} = \mathfrak{g}^{\wedge k}$.

□

Theorem 3.3 *(**Bianchi** II): Let $\mathfrak{g} = span\{e_1, e_2, e_3\}$ be a Lie algebra isomorphic to **Bianchi** II, given by the brackets $[e_1, e_2] = 0, [e_2, e_3] = e_1, [e_3, e_1] = 0$. Then,*

$$H_k^{Lie}(\mathfrak{g}) = \begin{cases} \mathbb{R}, & \text{if } k{=}0 \\ \langle e_2, e_3\rangle, & \text{if } k{=}1 \\ \langle e_1 \wedge e_2, e_1 \wedge e_3\rangle, & \text{if } k{=}2 \\ \langle e_1 \wedge e_2 \wedge e_3\rangle, & \text{if } k{=}3 \\ 0, & \text{if } k > 3 \end{cases}$$

Proof. The Chevalley-Eilenberg complex is reduced to

$$0 \xleftarrow{\partial_0} \mathbb{R} \xleftarrow{\partial_1} \mathfrak{g} \xleftarrow{\partial_2} \mathfrak{g}^{\wedge^2} \xleftarrow{\partial_3} \mathfrak{g}^{\wedge^3} \leftarrow 0$$

$\mathfrak{g} = span\{e_1, e_2, e_3\}$
$\mathfrak{g}^{\wedge 2} = span\{e_1 \wedge e_2, e_2 \wedge e_3, e_1 \wedge e_3\}$
$\mathfrak{g}^{\wedge 3} = span\{e_1 \wedge e_2 \wedge e_3\}$
$\mathfrak{g}^{\wedge 4} = 0$

$\partial_1(e_1) = 0,\ \partial_1(e_2) = 0,\ \partial_1(e_3) = 0$
$\partial_2(e_1 \wedge e_2) = [e_1, e_2] = 0$
$\partial_2(e_2 \wedge e_3) = [e_2, e_3] = e_1$
$\partial_2(e_1 \wedge e_3) = [e_1, e_3] = 0$

$$\begin{aligned}\partial_3(e_1 \wedge e_2 \wedge e_3) &= (-1)^4[e_1, e_2] \wedge e_3 + (-1)^5[e_1, e_3] \wedge e_2 + (-1)^6[e_2, e_3] \wedge e_1 \\ &= e_1 \wedge e_1 \\ &= 0\end{aligned}$$

$H_0^{Lie}(\mathfrak{g}) = \frac{ker\partial_0}{im\partial_1} = \frac{\mathbb{R}}{\langle 0\rangle} = \mathbb{R}$
$H_1^{Lie}(\mathfrak{g}) = \frac{ker\partial_1}{im\partial_2} = \frac{\langle e_1, e_2, e_3\rangle}{\langle e_1\rangle} = \langle e_2, e_3\rangle$
$H_2^{Lie}(\mathfrak{g}) = \frac{ker\partial_2}{im\partial_3} = \frac{\langle e_1 \wedge e_2, e_1 \wedge e_3\rangle}{\langle 0\rangle} = \langle e_1 \wedge e_2, e_1 \wedge e_3\rangle$
$H_3^{Lie}(\mathfrak{g}) = \frac{ker\partial_3}{im\partial_4} = \frac{\langle e_1 \wedge e_2 \wedge e_3\rangle}{\langle 0\rangle} = \langle e_1 \wedge e_2 \wedge e_3\rangle$ □

Theorem 3.4 *(**Bianchi** IV): Let $\mathfrak{g} = span\{e_1, e_2, e_3\}$ be a Lie algebra isomorphic to **Bianchi** IV, given by the brackets $[e_1, e_2] = 0, [e_2, e_3] = e_1 - e_2, [e_3, e_1] = e_1$. Then,*

$$H_k^{Lie}(\mathfrak{g}) = \begin{cases} \mathbb{R}, & \text{if } k{=}0 \\ \langle e_3\rangle, & \text{if } k{=}1 \\ 0, & \text{if } k > 1 \end{cases}$$

Proof. The Chevalley-Eilenberg complex is reduced to

$$0 \xleftarrow{\partial_0} \mathbb{R} \xleftarrow{\partial_1} \mathfrak{g} \xleftarrow{\partial_2} \mathfrak{g}^{\wedge^2} \xleftarrow{\partial_3} \mathfrak{g}^{\wedge^3} \leftarrow \cdots$$

$\mathfrak{g} = \{e_1, e_2, e_3\}$
$\mathfrak{g}^{\wedge 2} = \{e_1 \wedge e_2, e_2 \wedge e_3, e_1 \wedge e_3\}$
$\mathfrak{g}^{\wedge 3} = \{e_1 \wedge e_2 \wedge e_3\}$
$\mathfrak{g}^{\wedge 4} = 0$

$\partial_1(e_1) = 0$, $\partial_1(e_2) = 0$, $\partial_1(e_3) = 0$
$\partial_2(e_1 \wedge e_2) = [e_1, e_2] = 0$
$\partial_2(e_2 \wedge e_3) = [e_2, e_3] = e_1 - e_2$
$\partial_2(e_1 \wedge e_3) = [e_1, e_3] = -e_1$

$$\begin{aligned}\partial_3(e_1 \wedge e_2 \wedge e_3) &= (-1)^4[e_1, e_2] \wedge e_3 + (-1)^6[e_2, e_3] \wedge e_1 + (-1)^5[e_1, e_3] \wedge e_2 \\ &= (e_1 - e_2) \wedge e_1 + e_1 \wedge e_2 \\ &= e_1 \wedge e_1 - e_2 \wedge e_1 + e_1 \wedge e_2 \\ &= 2(e_1 \wedge e_2)\end{aligned}$$

$H_0^{Lie}(\mathfrak{g}) = \frac{ker\partial_0}{im\partial_1} = \frac{\mathbb{R}}{\langle 0\rangle} = \mathbb{R}$
$H_1^{Lie}(\mathfrak{g}) = \frac{ker\partial_1}{im\partial_2} = \frac{\langle e_1,e_2,e_3\rangle}{\langle e_1-e_2,-e_1\rangle} = \langle e_3\rangle$
$H_2^{Lie}(\mathfrak{g}) = \frac{ker\partial_2}{im\partial_3} = \frac{\langle e_1\wedge e_2\rangle}{\langle 2(e_1\wedge e_2)\rangle} = \langle 0\rangle$
$H_3^{Lie}(\mathfrak{g}) = \frac{ker\partial_3}{im\partial_4} = \frac{\langle 0\rangle}{\langle 0\rangle} = \langle 0\rangle$ □

Theorem 3.5 *(**Bianchi** V): Let $\mathfrak{g} = span\{e_1, e_2, e_3\}$ be a Lie algebra isomorphic to **Bianchi** V, given by the brackets $[e_1, e_2] = 0, [e_2, e_3] = e_2, [e_3, e_1] = e_1$. Then,*

$$H_k^{Lie}(\mathfrak{g}) = \begin{cases} \mathbb{R}, & \textit{if k=0} \\ \langle e_3\rangle, & \textit{if k=1} \\ \langle e_1 \wedge e_2\rangle, & \textit{if k=2} \\ \langle e_1 \wedge e_2 \wedge e_3\rangle, & \textit{if k=3} \\ \textit{0}, & \textit{if } k > 3 \end{cases}$$

Proof. The Chevalley-Eilenberg complex is reduced to
$0 \xleftarrow{\partial_0} \mathbb{R} \xleftarrow{\partial_1} \mathfrak{g} \xleftarrow{\partial_2} \mathfrak{g}^{\wedge^2} \xleftarrow{\partial_3} \mathfrak{g}^{\wedge^3} \leftarrow \cdots$

$\mathfrak{g} = \{e_1, e_2, e_3\}$
$\mathfrak{g}^{\wedge 2} = \{e_1 \wedge e_2, e_2 \wedge e_3, e_1 \wedge e_3\}$
$\mathfrak{g}^{\wedge 3} = \{e_1 \wedge e_2 \wedge e_3\}$
$\mathfrak{g}^{\wedge 4} = 0$

$\partial_1(e_1) = 0$, $\partial_1(e_2) = 0$, $\partial_1(e_3) = 0$
$\partial_2(e_1 \wedge e_2) = [e_1, e_2] = 0$
$\partial_2(e_2 \wedge e_3) = [e_2, e_3] = e_2$

$\partial_2(e_1 \wedge e_3) = [e_1, e_3] = -e_1$

$$\begin{aligned}\partial_3(e_1 \wedge e_2 \wedge e_3) &= (-1)^4[e_1, e_2] \wedge e_3 + (-1)^6[e_2, e_3] \wedge e_1 + (-1)^5[e_1, e_3] \wedge e_2 \\ &= e_2 \wedge e_1 + e_1 \wedge e_2 \\ &= 0\end{aligned}$$

$H_0^{Lie}(\mathfrak{g}) = \frac{ker\partial_0}{im\partial_1} = \frac{\mathbb{R}}{\langle 0\rangle} = \mathbb{R}$
$H_1^{Lie}(\mathfrak{g}) = \frac{ker\partial_1}{im\partial_2} = \frac{\langle e_1,e_2,e_3\rangle}{\langle e_2,-e_1\rangle} = \langle e_3\rangle$
$H_2^{Lie}(\mathfrak{g}) = \frac{ker\partial_2}{im\partial_3} = \frac{\langle e_1\wedge e_2\rangle}{\langle 0\rangle} = \langle e_1 \wedge e_2\rangle$
$H_3^{Lie}(\mathfrak{g}) = \frac{ker\partial_3}{im\partial_4} = \frac{\langle e_1\wedge e_2\wedge e_3\rangle}{\langle 0\rangle} = \langle e_1 \wedge e_2 \wedge e_3\rangle$ □

Theorem 3.6 *(**Bianchi** VI_h): Let $\mathfrak{g} = span\{e_1, e_2, e_3\}$ be a Lie algebra isomorphic to **Bianchi** VI_h, given by the brackets $h \le 0, [e_1, e_2] = 0, [e_2, e_3] = e_1 - he_2, [e_3, e_1] = he_1 - e_2$ Then,*

$$H_k^{Lie}(\mathfrak{g}) = \begin{cases} \mathbb{R}, & \textit{if k=0} \\ \frac{\langle e_1,e_2,e_3\rangle}{\langle e_1+e_2\rangle}, & \textit{if k=1 and } h=-1 \\ \langle e_3\rangle, & \textit{if k=1 and } h \neq -1 \\ \langle e_1 \wedge e_3 - e_2 \wedge e_3\rangle, & \textit{if k=2 and } h=-1 \\ \langle e_1 \wedge e_2\rangle, & \textit{if k=2 and } h=0 \\ \langle 0\rangle, & \textit{if k=2 and } h \neq -1 \textit{ and } h \neq 0 \\ \langle e_1 \wedge e_2 \wedge e_3\rangle, & \textit{if k=3 and } h=0 \\ \textit{0}, & \textit{else.} \end{cases}$$

Proof. The Chevalley-Eilenberg complex is reduced to
$0 \xleftarrow{\partial_0} \mathbb{R} \xleftarrow{\partial_1} \mathfrak{g} \xleftarrow{\partial_2} \mathfrak{g}^{\wedge^2} \xleftarrow{\partial_3} \mathfrak{g}^{\wedge^3} \leftarrow \cdots$

$\mathfrak{g} = \{e_1, e_2, e_3\}$
$\mathfrak{g}^{\wedge 2} = \{e_1 \wedge e_2, e_2 \wedge e_3, e_1 \wedge e_3\}$
$\mathfrak{g}^{\wedge 3} = \{e_1 \wedge e_2 \wedge e_3\}$
$\mathfrak{g}^{\wedge 4} = 0$

$\partial_1(e_1) = 0$, $\partial_1(e_2) = 0$, $\partial_1(e_3) = 0$
$\partial_2(e_1 \wedge e_2) = [e_1, e_2] = 0$
$\partial_2(e_2 \wedge e_3) = [e_2, e_3] = e_1 - he_2$
$\partial_2(e_1 \wedge e_3) = [e_1, e_3] = -he_1 + e_2$

$$\begin{aligned}\partial_3(e_1 \wedge e_2 \wedge e_3) &= (-1)^4[e_1, e_2] \wedge e_3 + (-1)^6[e_2, e_3] \wedge e_1 + (-1)^5[e_1, e_3] \wedge e_2 \\ &= (e_1 - he_2) \wedge e_1 + (he_1 - e_2) \wedge e_2 \\ &= e_1 \wedge e_1 - he_2 \wedge e_1 + he_1 \wedge e_2 - e_2 \wedge e_2 \\ &= 2h(e_1 \wedge e_2)\end{aligned}$$

Note that $ker\partial_2 = \langle e_1 \wedge e_2, e_1 \wedge e_3 - e_2 \wedge e_3 \rangle$ if $h = -1$, and $ker\partial_2 = \langle e_1 \wedge e_2 \rangle$ if $h = 0$. Also, $ker\partial_3 = \langle e_1 \wedge e_2 \wedge e_3 \rangle$ if $h = 0$, and $ker\partial_2 = \langle 0 \rangle$ if $h \neq 0$. We then have

$H_0^{Lie}(\mathfrak{g}) = \frac{ker\partial_0}{im\partial_1} = \frac{\mathbb{R}}{\langle 0 \rangle} = \mathbb{R}$

$$H_1^{Lie}(\mathfrak{g}) = \frac{ker\partial_1}{im\partial_2} = \frac{\langle e_1, e_2, e_3 \rangle}{\langle e_1 - he_2, -he_1 + e_2 \rangle} = \begin{cases} \frac{\langle e_1, e_2, e_3 \rangle}{\langle e_1 + e_2 \rangle}, & \text{if } h = -1 \\ \langle e_3 \rangle, & \text{if } h \neq -1. \end{cases}$$

$$H_2^{Lie}(\mathfrak{g}) = \frac{ker\partial_2}{im\partial_3} = \frac{ker\partial_2}{\langle 2h(e_1 \wedge e_2) \rangle} = \begin{cases} \langle e_1 \wedge e_3 - e_2 \wedge e_3 \rangle, & \text{if } h = -1 \\ \langle e_1 \wedge e_2 \rangle, & \text{if } h = 0 \\ \langle 0 \rangle, & \text{if } h \neq -1, 0 \end{cases}$$

$$H_3^{Lie}(\mathfrak{g}) = \frac{ker\partial_3}{im\partial_4} = \frac{ker\partial_3}{\langle 0 \rangle} = \begin{cases} \langle e_1 \wedge e_2 \wedge e_3 \rangle, & \text{if } h = 0 \\ 0, & \text{if } h \neq 0. \end{cases}$$ □

Theorem 3.7 *(**Bianchi** VII_h): Let $\mathfrak{g} = span\{e_1, e_2, e_3\}$ be a Lie algebra isomorphic to **Bianchi** VII_h, given by the brackets $h \geq 0, [e_1, e_2] = 0, [e_2, e_3] = e_1 - he_2, [e_3, e_1] = he_1 + e_2$ Then,*

$$H_k^{Lie}(\mathfrak{g}) = \begin{cases} \mathbb{R}, & \text{if } k=0 \\ \langle e_3 \rangle, & \text{if } k=1 \\ \langle e_1 \wedge e_2 \rangle, & \text{if } k=2 \text{ and } h = 0 \\ 0, & \text{if } k=2 \text{ and } h \neq 0 \\ \langle e_1 \wedge e_2 \wedge e_3 \rangle, & \text{if } k=3 \text{ and } h = 0 \\ 0, & \text{else.} \end{cases}$$

Proof. The Chevalley-Eilenberg complex is reduced to

$$0 \xleftarrow{\partial_0} \mathbb{R} \xleftarrow{\partial_1} \mathfrak{g} \xleftarrow{\partial_2} \mathfrak{g}^{\wedge^2} \xleftarrow{\partial_3} \mathfrak{g}^{\wedge^3} \leftarrow \cdots$$

$\mathfrak{g} = \{e_1, e_2, e_3\}$
$\mathfrak{g}^{\wedge 2} = \{e_1 \wedge e_2, e_2 \wedge e_3, e_1 \wedge e_3\}$
$\mathfrak{g}^{\wedge 3} = \{e_1 \wedge e_2 \wedge e_3\}$
$\mathfrak{g}^{\wedge 4} = 0$

$\partial_1(e_1) = 0$, $\partial_1(e_2) = 0$, $\partial_1(e_3) = 0$
$\partial_2(e_1 \wedge e_2) = [e_1, e_2] = 0$
$\partial_2(e_2 \wedge e_3) = [e_2, e_3] = e_1 - he_2$
$\partial_2(e_1 \wedge e_3) = [e_1, e_3] = -he_1 - e_2$

$$\begin{aligned} \partial_3(e_1 \wedge e_2 \wedge e_3) &= (-1)^4[e_1, e_2] \wedge e_3 + (-1)^6[e_2, e_3] \wedge e_1 + (-1)^5[e_1, e_3] \wedge e_2 \\ &= (e_1 - he_2) \wedge e_1 + (he_1 + e_2) \wedge e_2 \\ &= e_1 \wedge e_1 - he_2 \wedge e_1 + he_1 \wedge e_2 + e_2 \wedge e_2 \\ &= 2h(e_1 \wedge e_2) \end{aligned}$$

Note that $ker\partial_3 = \langle e_1 \wedge e_2 \wedge e_3 \rangle$ if $h = 0$, and $ker\partial_3 = \langle 0 \rangle$ if $h \neq 0$. We then have

$H_0^{Lie}(\mathfrak{g}) = \frac{ker\partial_0}{im\partial_1} = \frac{\mathbb{R}}{\langle 0 \rangle} = \mathbb{R}$

$H_1^{Lie}(\mathfrak{g}) = \frac{ker\partial_1}{im\partial_2} = \frac{\langle e_1, e_2, e_3 \rangle}{\langle e_1 - he_2, -he_1 - e_2 \rangle} = \langle e_3 \rangle$

$H_2^{Lie}(\mathfrak{g}) = \frac{ker\partial_2}{im\partial_3} = \frac{\langle e_1 \wedge e_2 \rangle}{\langle 2h(e_1 \wedge e_2) \rangle} = \begin{cases} \langle e_1 \wedge e_2 \rangle, & \text{if } h = 0 \\ 0, & \text{if } h \neq 0 \end{cases}$

$H_3^{Lie}(\mathfrak{g}) = \frac{ker\partial_3}{im\partial_4} = \frac{ker\partial_3}{\langle 0 \rangle} = \begin{cases} \langle e_1 \wedge e_2 \wedge e_3 \rangle, & \text{if } h = 0 \\ 0, & \text{else.} \end{cases}$ □

Theorem 3.8 *(**Bianchi** VIII): Let $\mathfrak{g} = span\{e_1, e_2, e_3\}$ be a Lie algebra isomorphic to **Bianchi** VIII, given by the brackets $[e_1, e_2] = -e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$ Then,*

$$H_k^{Lie}(\mathfrak{g}) = \begin{cases} \mathbb{R}, & \text{if } k=0 \\ \langle e_1 \wedge e_2 \wedge e_3 \rangle, & \text{if } k=3 \\ 0, & \text{if } k > 0 \text{ and } k \neq 3 \end{cases}$$

Proof. The Chevalley-Eilenberg complex is reduced to

$0 \xleftarrow{\partial_0} \mathbb{R} \xleftarrow{\partial_1} \mathfrak{g} \xleftarrow{\partial_2} \mathfrak{g}^{\wedge^2} \xleftarrow{\partial_3} \mathfrak{g}^{\wedge^3} \leftarrow \cdots$

$\mathfrak{g} = \{e_1, e_2, e_3\}$
$\mathfrak{g}^{\wedge 2} = \{e_1 \wedge e_2, e_2 \wedge e_3, e_1 \wedge e_3\}$
$\mathfrak{g}^{\wedge 3} = \{e_1 \wedge e_2 \wedge e_3\}$
$\mathfrak{g}^{\wedge 4} = 0$

$\partial_1(e_1) = 0$, $\partial_1(e_2) = 0$, $\partial_1(e_3) = 0$
$\partial_2(e_1 \wedge e_2) = [e_1, e_2] = -e_3$
$\partial_2(e_2 \wedge e_3) = [e_2, e_3] = e_1$
$\partial_2(e_1 \wedge e_3) = [e_1, e_3] = -e_2$

$$\begin{aligned} \partial_3(e_1 \wedge e_2 \wedge e_3) &= (-1)^4[e_1, e_2] \wedge e_3 + (-1)^6[e_2, e_3] \wedge e_1 + (-1)^5[e_1, e_3] \wedge e_2 \\ &= -e_3 \wedge e_3 + e_1 \wedge e_1 + e_2 \wedge e_2 \\ &= 0 \end{aligned}$$

$H_0^{Lie}(\mathfrak{g}) = \frac{ker\partial_0}{im\partial_1} = \frac{\mathbb{R}}{\langle 0 \rangle} = \mathbb{R}$

$H_1^{Lie}(\mathfrak{g}) = \frac{ker\partial_1}{im\partial_2} = \frac{\langle e_1, e_2, e_3 \rangle}{\langle -e_3, e_1, -e_2 \rangle} = \langle 0 \rangle$

$H_2^{Lie}(\mathfrak{g}) = \frac{ker\partial_2}{im\partial_3} = \frac{\langle 0 \rangle}{\langle 0 \rangle} = \langle 0 \rangle$

$H_3^{Lie}(\mathfrak{g}) = \frac{ker\partial_3}{im\partial_4} = \frac{\langle e_1 \wedge e_2 \wedge e_3 \rangle}{\langle 0 \rangle} = \langle e_1 \wedge e_2 \wedge e_3 \rangle$ □

Theorem 3.9 *(**Bianchi** IX): Let $\mathfrak{g} = span\{e_1, e_2, e_3\}$ be a Lie algebra isomorphic to **Bianchi** IX, given by the brackets $[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$ Then,*

$$H_k^{Lie}(\mathfrak{g}) = \begin{cases} \mathbb{R}, & \text{if } k=0 \\ \langle e_1 \wedge e_2 \wedge e_3 \rangle, & \text{if } k=3 \\ 0, & \text{if } k > 0 \text{ and } k \neq 3 \end{cases}$$

Proof. The Chevalley-Eilenberg complex is reduced to

$0 \xleftarrow{\partial_0} \mathbb{R} \xleftarrow{\partial_1} \mathfrak{g} \xleftarrow{\partial_2} \mathfrak{g}^{\wedge^2} \xleftarrow{\partial_3} \mathfrak{g}^{\wedge^3} \leftarrow \cdots$

$\mathfrak{g} = \{e_1, e_2, e_3\}$
$\mathfrak{g}^{\wedge 2} = \{e_1 \wedge e_2, e_2 \wedge e_3, e_1 \wedge e_3\}$
$\mathfrak{g}^{\wedge 3} = \{e_1 \wedge e_2 \wedge e_3\}$
$\mathfrak{g}^{\wedge 4} = 0$

$\partial_1(e_1) = 0$, $\partial_1(e_2) = 0$, $\partial_1(e_3) = 0$
$\partial_2(e_1 \wedge e_2) = [e_1, e_2] = e_3$
$\partial_2(e_2 \wedge e_3) = [e_2, e_3] = e_1$
$\partial_2(e_1 \wedge e_3) = [e_1, e_3] = -e_2$

$$\begin{aligned} \partial_3(e_1 \wedge e_2 \wedge e_3) &= (-1)^4[e_1, e_2] \wedge e_3 + (-1)^6[e_2, e_3] \wedge e_1 + (-1)^5[e_1, e_3] \wedge e_2 \\ &= e_3 \wedge e_3 + e_1 \wedge e_1 + e_2 \wedge e_2 \\ &= 0 \end{aligned}$$

$H_0^{Lie}(\mathfrak{g}) = \frac{ker\partial_0}{im\partial_1} = \frac{\mathbb{R}}{\langle 0 \rangle} = \mathbb{R}$
$H_1^{Lie}(\mathfrak{g}) = \frac{ker\partial_1}{im\partial_2} = \frac{\langle e_1, e_2, e_3 \rangle}{\langle e_3, e_1, -e_2 \rangle} = \langle 0 \rangle$
$H_2^{Lie}(\mathfrak{g}) = \frac{ker\partial_2}{im\partial_3} = \frac{\langle 0 \rangle}{\langle 0 \rangle} = \langle 0 \rangle$
$H_3^{Lie}(\mathfrak{g}) = \frac{ker\partial_3}{im\partial_4} = \frac{\langle e_1 \wedge e_2 \wedge e_3 \rangle}{\langle 0 \rangle} = \langle e_1 \wedge e_2 \wedge e_3 \rangle$ □

Remark 3.10 *From the theorems above, one notices that the Lie algebras **Bianchi** V, **Bianchi** VI_h and **Bianchi** VII_h have the same homology when $h = 0$, **Bianchi** IV, and **Bianchi** VII_h have the same homology when $h \neq 0$, and the Lie algebras **Bianchi** VIII and **Bianchi** IX have the same homology.*

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