How Conceptual Understanding Can Improve Mathematical Intuition

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Abstract

This is a study that was conducted on 61 high school sophomore and juniors to see if training their conceptual understanding of exponential growth and decay improves their intuition of the concept. The students were given an anonymous short four question pre-test followed by a lecture followed by an identical post-test. The students were not allowed to use calculators or refer to their first test when taking the second. The lecture trained students on the behavior of exponential growth and decay. Statistical analysis indicates that there was improvement from the pre-test to the post-test. Other research indicates that training intuition is critical for improving mathematical understanding.

Question

In this study, the researchers wanted to see if there exists a correlation between training a mathematical concept and an improvement in intuition for that concept.

Purpose and Motivation

The purpose of this study is to evaluate whether or not training a student's conceptual understanding of a mathematical concept improves their mathematical intuition.

Studies indicate that humans and animals share an innate sense of numbers. Human infants and animals have a concept of quantity and have the ability to compare sizes. A squirrel knows that it should grab the pile of 10 acorns instead of the pile of 5. While this ability isn't precise, it is notable that even 6-month olds can distinguish between 6 and 18 dots but not 6 and 12 dots (Xu and Spelke, 2000). This understanding of numbers is called the approximate number system or ANS. It is understood that this is an understanding that humans are born with. In adults, an example of one using their ANS would be when you go to the grocery store and it is time to check out. You observe the check out lanes and, without counting, you decide which line is shortest and moving the quickest. Research shows that a person's ANS undergoes critical developments from infancy to adulthood, sharpening the precision of his or her approximations. The ANS continues to improve as a person gets older until about the age of 30. At this age, the acuity of the ANS begins to steadily weaken for the rest of their life (Halberda, Ly, Wilmer, Naiman, & Germine, 2012).

There is a wide range of mathematical ability at all levels. Some students require more attention and explanation. Others need little guidance and can understand mathematical concepts with little training. Perhaps, the students who require less guidance have a better intuition for how numbers work and for mathematics in general. If teachers emphasized training mathematical intuition instead of focusing on training students' ability to memorize and compute, would students be more successful in mathematics? Many students will reach a point in solving a problem where they are unsure of how to move forward. If this particular point was recognized, and they were pushed in the right direction, they might have a notion as to how the problem should be continued. For example, a student is computing a long division problem, and he or she forgets that the next step is to subtract their product from the corresponding digits in the divisor. The teacher might give a hint such as, "the dividend didn't go into those digits of the divisor evenly, so what should you do to find the difference." This would require the student to think about why the next step is necessary as opposed to just following a process.

Conceptualizing mathematical abstractions might help improve students' intuition. Intuition can be defined the power or faculty of attaining to direct knowledge or cognition without evident rational thought and inference (Merriam-Webster Dictionary). Students are often trained on how to do a mathematical procedure without understanding why they are doing each step. This means that they do not understand the mathematical concept and are following a routine. For example, a first grader might be learning multiple-digit addition. He or she is told to enter 100 + 20 in their calculator. The student makes a mistake and enters 100+200, which sums to 300. If the student doesn't understand the concept of addition, he or she might not recognize that the answer 300 is bizarre and far from the correct answer of 120. He or she would be trained on the procedure of typing in two addends and receiving a sum, but might not understand the operation of addition.

Improving mathematical intuition might be the missing puzzle piece in math education. The lack of this training might be causing so many students to fall behind their state's standards. In the spring of 2014, 77.4% of students taking either Geometry, Algebra 2, or Statistics in Georgia did not meet the standards for the End Of Course Tests. This is a significant distinction from other subjects such as American Literature, which only had a 7.5% failure rate, United States History with a 27.2% failure rate, 9th grade Literature and Composition with a 12.3% failure rate, and Biology with a 25% failure rate (EOCT Statewide Scores). These statistics are concerning, and it is possible that a lack of conceptual understanding and mathematical intuition is partly to blame.

If it were concluded that conceptual understanding does not improve mathematical intuition, mathematics education should be approached in a different way. Perhaps there is another reason why students are suffering in mathematics that is different from a lack of mathematical intuition. Researchers might look into improving curriculum, teacher education requirements, being more selective in hiring math teachers, etc. This would need to be explored to understand the root causes of the wide range of mathematical ability.

Procedure

The researcher had the opportunity to observe and student teach sophomore and junior high school students at public schools in Milledgeville, Georgia. During the spring of 2015, the researcher conducted a study by carefully preparing a short test and a lesson plan to observe the effect of emphasizing conceptual understanding on mathematical intuition.

Procedure

A lecture was planned for a new topic that the students were supposed to be learning according to their state standards. The topic was exponential growth and decay. The lecturer prepared a short test consisting of four questions on this topic. The answer selections were multiple choice and consisted of a range that they were to select that they thought their answer would fall under, i.e. 0-100, 101-1000, 1001-10000, etc. [See Appendix I]. The test was anonymous, asked for gender, grade level, and if the student had previously taken that course. Before the lecture, the students were asked to take the test and to try their best to estimate where their answer would land. They were not allowed to use a calculator and were encouraged to estimate instead of calculate. Once the students were finished, they flipped their tests over and the lecture began. After the lecture, the students were given the same test as before with the same instructions. In addition, they were not allowed to reference their first test. After the students were finished, the pre and post tests were attached and collected. This procedure was conducted for three different class periods for a total of 61 students.

Lecture Design and Overview

The lecture was designed so as to not directly prepare for the test. It was a inteded to improve intuition by training conceptual understanding as opposed to training intuition by experience. The lecture begun with the students stating differences between linear and exponential functions. Students responded with answers such as, "linear graphs are straight while exponential graphs are curved." When prompted why this is the case, students were able to arrive at the conclusion that exponential graphs increase or decrease rapidly while linear graphs increase or decrease steadily. The answers to these questions gave the lecturer a basis for the students understanding of exponential behavior. The students were then asked a question that tested their ability to estimate the size of a quantity that grows exponentially. The question was the classic chess board problem:

A king asked one of his noble knights what reward he would like to receive for an honorable service. The knight said he would like to place a grain of rice on the first square of a chess board and he'd like to double the amount of rice from each previous square on to the next square. The king agreed to fulfill the knight's request. How much rice did he owe the knight?

Some students responded in the hundreds, some in the thousands, some in the millions. After guiding the students to realize that for any n square, the amount of grains of rice equated to 2^n, they figured out that this number was much larger than they had previously estimated and was in the ballpark of 9 quintillion. It was important to train the students to understand how rapidly numbers increase or decrease when working with exponential growth or decay, because this particular concept is counterintuitive. It was intended for the students to develop an intuition for the behavior of exponents. The students were taught the exponential growth and decay equations: $y = a(1+r)^t$ and $y = a(1-r)^t$ respectively. They used their calculators to solve problems that asked for the amount of growth or decay y, given the starting amount a, the rate of increase or decrease r and the period of time t. The students needed to have a mechanical understanding of the exponential growth and decay process in order to solidify the concept. The students were asked to put their calculators away and were asked similar questions. They were to give a ballpark range of where they thought the answers fell and were given feedback on the accuracy of their estimates. Throughout the lecture, it was important that the lecturer not give answers. The students were guided in the direction that they were to develop an intuition for. The students received explanations for answers and were shown how to derive answers on their own. This strategy was crucial in ensuring that students were getting a thorough understanding of the exponential concepts that were previously foreign. The lecture ended with a similar question to the first question asked. It forced students to reflect on the concept, as opposed to mechanically solving the problem. Each lecture lasted approximately one hour.

Data Collection and Analysis

Once the tests were collected, each test was graded with a score of correct answers out of four. The students' first and second attempt scores were compared. Some notable trends were that 41% of students' scores improved on the second test, 34% of students' scores did not change, and 25% of students' scores worsened. 10 of the 61 students had previously taken the course, and 5 of those students' scores improved on the second test. After looking at the obvious trends, statistical analysis was conducted to find a confidence score that would suggest that improvement actually occurred after taking the second test.

Procedure for Statistical Analysis

Each student's first and second test scores, s_b and s_a were recorded and the averages of those were taken. The variables μ_b and μ_a represent the averages for the test taken before the lecture and the test taken after the lecture respectively.

$\mu_b = .45492$

$\mu_a = .53279$

$\Delta\mu = .53279 - .45492 = .07787$

The standard deviations σ_b and σ_a were calculated.

$\sigma_b = .24370$

$\sigma_a = .29863$

The variance for each student was calculated, then the variance of all students was calculated.

$\sigma^2_{\Delta\mu} = \frac{1}{N}\sum_{n=1}^{61} \sigma^2_{s_b} + \sigma^2_{s_a} = .00144$, where $N = 61$ students

$\sqrt{\sigma^2_{\Delta\mu}} = \sigma_{\Delta\mu} = .03795$

Using a z-score of $z_{.05} = 1.645$,

$.07787 - (1.645 * .03795) = .01544$, which indicates that with a confidence score of 95%, there was at least .01544 improvement.

Assuming the null hypothesis, $\frac{.07787-0}{.03795} = 2.0519$

A z-score of 2.0519 indicates that the researchers can say with 98% confidence, improvement occurred between the first and second quiz.

Other Results and Similar Findings

In order to observe a link between ANS and mathematical ability, one might look to see if a person who lacks in one skill also lacks in the other. People who suffer form dyscalculia have difficulty understanding numbers, using numbers, and making arithmetical calculations. People of all IQ ranges and people who have been exposed to exceptional mathematical education can suffer from dyscalculia. This disability affects an estimated 6-10% of the population (Butterworth, 2010). It has been noted that people with dyscalculia also have an inaccurate sense of nonsymbolic numerical approximations as compared to people of the same age with similar cognitive abilities (Fiegenson, Libertus, & Halberda, 2013). The correlation between dyscalculic people and weaker ANS suggests that mathematical ability and ANS might also be correlated.

When observing people who are developing typically for their age, there has also been a correlation between ANS and formal mathematical ability. In a study, a group of 14-year-olds were presented with a psychosocial modeling to gage the sharpness of their ANS. The students showed a wide range of ANS ability, and it directly correlated with formal mathematical ability (Halberda, Mazzacco, & Fiegenson, 2008). A similar study showed that the same correlation exists for adults who have completed their formal mathematics training (De Wind & Brannon, 2012; Libertus, Odic, & Halberda, 2012).

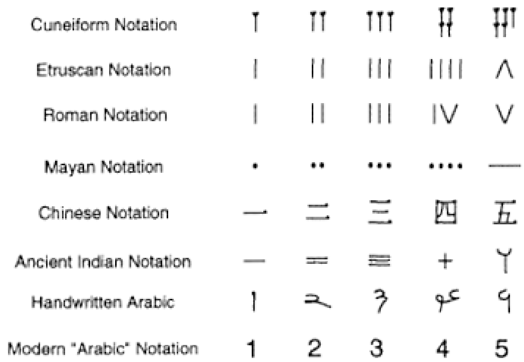
It has been found that in addition to the ANS, humans have access to an ENS, an exact number system. The ENS is trained rather than instinctual. This set of abilities would consist of computational skills and calculations. This skill is essential for learning mathematics in school. The ENS is unique to humans and is developed throughout childhood and into early adulthood. The acuity of the ENS is linked with the development in understanding of language. They go hand in hand with each other. It has been noted that communities with poor verbal counting routines also have poor exact number concepts (Libertus, 2015). One study observed children between first and sixth grades, and measured each student's ANS and ENS. The ANS was measured in multiple ways by comparing quantities using different representations. The ENS was measured by giving students different math facts of different complexities. In the results, all measures of the ANS strongly correlated with ENS ability (Chagas, Wood, Knops, Krinzinger, Lonnemann, Starling-Alves, Willmes, Haase, 2014).

The question that these findings pose is whether or not if one of these number systems is trained and improved, does the aptitude for the other improve? Researchers showed that after working on non-symbolic addition and subtraction exercises, which improves ANS acuity, adults yielded much better results on an exact arithmetic test than adults who did not train their ANS (Parks and Brannon, 2013). Using the ANS is one way of using mathematical intuition. Sharpening the ANS, the ability to conceptualize, enhances one's intuition. As seen in this case, improving the ANS improved computational performance and understanding. This indicates that improving the ability to comprehend mathematical concepts improves understanding without conscious reasoning.

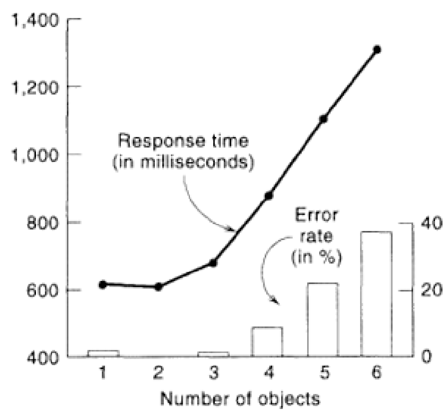
Another group of researchers did a study on a class of 8th graders to observe the relationships between number sense and understanding of exponential concepts. To measure each student's understanding, the researchers used an exponential comparison test and a number sense scale for exponents. The test contained 20 items of different levels of difficulty, in which students were to compare to quantities represented in exponential form. The exponential number sense scale was developed using the components that describe number sense in the relevant literature. These consist of equivalent expression, number estimation, number value, and operation effect. The students were given the test in an interview format by researchers. They were told to explain their thought processes out loud and were asked questions to understand their thought processes. The researchers used qualitative techniques to analyze the data. The results yielded that students had a difficult time using number sense to answer questions about exponential concepts (Iymen & Duatepe-Paksu, 2015). There is a disconnect between the ANS and understanding this abstract concept. Perhaps, this is because of the counter intuitiveness of exponential concepts.

Adults and Large Numbers

Many people probably wonder how the Roman numeral system was developed. Why is it that the first three numbers I, II, and III seem so intuitive and everything after that seems arbitrary? The representations for one, two, and three make sense because they contain the amount of vertical lines as numbers being represented. It turns out, that many other numerical notation systems have the same structure, where the first three numbers are represented as one might intuit and the rest are arbitrary.



Studies show that the limit of three numbers for visual representations isn't so arbitrary. In the early 1900s, Bertrand Bourdon conducted an experiment with himself as the subject to test how quickly he could state the number of dots arranged horizontally when they appeared before him. With almost perfect precision and very little time, he could say aloud the numbers one, two, and three when they appeared before him. After three, the amount of time it took to register the numbers and the number of errors per set of dots rapidly increased (Dehaene, 1997).



Humans are also able to approximate quantities when given a visual representation. However, it has been noted that the smaller the number, the more accurate our approximations. For example, people can more accurately distinguish 10 dots from 20 dots than 90 dots from 100 dots. As numbers get larger, they become more of an abstraction for humans. Our "mental ruler" isn't evenly spaced. As numbers get larger, our tick marks get closer together. Our approximation accuracy decreases as numbers grow. The compression is similar to a logarithmic scale. Because of this, humans are biased towards smaller quantities. Humans are more familiar with smaller numbers. As an example, when people are asked to randomly generate numbers in a given range, they tend to give smaller numbers than larger numbers (Dehaene, 1997).

Series A:	879	5	1,322	1,987	212	1,776	1,561	437	1,098	663
Series B:	238	5	689	1,987	16	1,446	1,018	58	421	117

When presented these two sets of numbers that range between 1 and 2,000, people more often choose series B to appear more random when in actuality, series A samples the set more evenly. Series B better fits a person's mental number line, thus many people would favor this set. In order to recognize that series A is more random, one's numerical intuition must be trained (Dehaene, 1997).

Implications for Education

George Polya says that the ability to estimate or predict a solution may enhance a person's learning motivation. In order to develop the appropriate thinking processes that lead to precise estimation and approximation, formal knowledge is necessary as well as intuitive knowledge. Teaching subjects such as squares, roots, and other mathematical topics should combine estimations kills with intuitive and creative elements (Patkin & Gazit, 2013). Jeremy Bruner claims that the limited attention paid to the development of intuitive thinking attributes to the decreased comprehension in mathematics. Intuition and analytical thought processes accompany each other, and when used together they create different thought processes. Intuitive thinking is characterized by immediate reactions and of skipping steps. Intuition involves unconscious "preparation and illumination" (Giardano, 2010). The mind prepares a result and then the person's conscious understands the result and sees it as a solution. Dehaene's argument is similar. She says that the intuitive operation consists of three elements, in which the word can be defined: it is fast, automatic and inaccessable to introspection. Humans aren't able to reflect on the root of their intuitive capabilities, but they are able to train their intuition. Meaning that one can never have a perfect intuition but that it can be continually sharpened. Elijah Chudnoff describes mathematical intuition as an experience that is similar to sensory perception because it gives its subjects non-inferential access to a world of facts. In the mathematics world, however, the facts are much more abstract, thus requiring more training (Chudnoff, 2014).

Concluding Remarks

Limitations

This research should be perceived as preliminary, as it is limited in a few ways. The sample size was relatively small, containing only 61 students. The students represented one high school in rural Georgia. One of the classes had a 30-minute lunch break during the period, thus the material discussed before lunch may not have been fresh in mind after coming back. The students may have also been eager to leave for lunch and might not have been as attentive during that portion of the lecture. The lecture consisted of one particular topic and was given to one age group of students. It was also difficult to have a control group in this study. The pre-test that the students took was used as the control. In addition, the test was created by hand, therefore, the ordering of questions and answers may have been biased. When reviewing some tests, the researchers noticed that some students attempted doing some calculations. It would be favorable to prevent any calculation. The students answers were only graded as right or wrong, and there was no investigation as to the though process behind incorrect answers.

Suggestions for Improved Further Research

The study could be improved and extended by eliminating these restraints. If this study sampled more students, different age groups, and different concepts, the research would be more solidified and useful. If the researchers underwent formal training on language and strategy on improving intuition, the approach may be more successful. It would be desirable to understand how to more fully isolate the mathematical concepts from the intuitive process. Also, a computer based test with time constraints could encourage using the intuitive process as opposed to doing raw calculations. It could also be helpful to calculate the percentages of over and under estimation, and determine how far off students answers were from the correct one. The research conducted should be used as an idea or example for further research.

Results and Conclusions

This research shows that training conceptual understanding of a mathematical concept likely improves mathematical intuition. The researchers were able to use statistical analysis to conclude that there was at least .01544 improvement with 95% confidence, and that some degree of improvement occurred with 98% confidence. In the evaluation of this study, the researchers considered any improvement to be significant as this suggests the hypothesis is true to some extent. The behavior of exponential growth and decay was mostly foreign to the students, prior to this study. After training their conceptual understanding, they had an improved intuition of the concept. The study indicates that the hypothesis is likely true if extended to a more representative sample. Students who are behind in mathematics may be able to improve their understanding and capabilities if there is a focus on improving intuition instead of computational procedures.

Appendix I

[Sample Test]

Do not write your name on this test.

Gender: __________ Grade level: __________

Is this your first time taking this class? Y or N

Directions: For each of these questions, select the range in which you think the answer lies in.

1. Your parents decide to give you an allowance for 10 days. They give you $2 on the first day, $4 on the second day, and they continue to give you double the amount of money from the previous day. How much money would you receive on the 10th day?

a) 0-100
b) 101-1,000
c) 1,001-10,000
d) 10,001-100,000
e) 100,001-1,000,000

2. If you have a dollar and fold it in half, the thickness of the dollar doubles. If you fold the dollar in half 8 times, how many times thicker will the dollar be once you have finished folding?

a) 0-100
b) 101-1,000
c) 1,001-10,000
d) 10,001-100,000
e) 100,001-1,000,000

3. The population of a city started at 1,000 people. The population grew at a rate of 30% per year for 15 years. What is the population of the city after 15 years?

a) 0-100
b) 101-1,000
c) 1,001-10,000
d) 10,001-100,000
e) 100,001-1,000,000

4. The tuition to go to a university starts at $4,300. Tuition increased by 5% each year. How much does it cost to go to this university after 20 years?

a) 0-100
b) 101-1,000
c) 1,001-10,000
d) 10,001-100,000
e) 100,001-1,000,000

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