The JAM Derivative

An Exploration of Function-Order Derivatives

Matthew
Pearson

12/13/2016

Abstract

In Quantum Mechanics the derivative is an anti-hermitian operator in Hilbert Space. In this space this operator can be represented by an infinite dimensional matrix. Considering this, some questions were raised about what could be done with this matrix representation. Letting D represent the derivative operator, we have that $DD = D^2$ represents the second order derivative operator, and thusly D^n represents the nth order derivative operator. Non-integers orders have been considered since the beginning of Calculus. With these defined, it is interesting to consider non-constant orders of differentiation. Letting $f(x)$ be some function, it is interesting to consider the case $D^{f(x)}$, the case of an operator which acts as a derivative of order $f(x)$. Since Quantum Mechanics deals with state vecotrs in a complex space, we shall define this operator using fourier transforms on functions expanded into fourier series. We thus have an an operator from the set of complex functions to itself with utility for any periodic or bounded real or complex function. We notate this new operator J^{α_m}, since within these parameters the operator acts as a multi-ordered variant of the integral operator.

1 Introduction

The goal of this research is the expansion of the derivative definition. We use a model of the derivative where we can represent it as an operator in a physical Hilbert space. From here we imagine the matrix representation of this operator where the order of the derivative is the power in which the matrix representation is raised. From this perspective it is clear that the power that this matrix is raised to can be non-integer, perhaps even be a matrix or function power. Well this implies the thought of a function order derivative. That is the inspiration of this research. How and in what way can a derivative, where its order is a real function, be defined? This definition would have to have the same result as an integer order derivative for its own integer order. There would be a need of a definition for any real-number order derivative to obtain a definition for real functions. So it was first necessary to do some research on the field of fractional derivatives to see where to go from there.

In beginning this research, several sources were used to provide an introduction into Fractional Calculus and Fourier Analysis. For fractional calculus references I found Adam Loverro's "Fraction Calculus: History, Definitions and Applications for the Engineer" and University of California-Davis's John Hunter and Bruno Nachtergale's notes on Apllied analysis, very helpful influential on my research in Fractional Calculus. Simultaneously, Javier Duoandikoetxea's "Fourier Analyis" and Rajendra Bhatia's "Fourier Series" where helpful primers on Fourier Analysis. What this research amounted to was the concept of defining a fractional derivative in terms of Fourier Series. Looking at the "classical" fractional derivative definition from Loverro:

$$D^{\alpha} f(t) := \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau,\ m-1<\alpha<m \qquad [1]$$

We find that this definition has a lot of obstacles when it comes to defining a derivative of function order. Any function not bounded on the y-axis by an interval of 1, which accounts for the majority of functions, makes m not well defined. Furthermore, it generates an integral that would likely be extremely difficult, if not impossible. So we shift our attention to the Fourier Transform in order to define a Fractional derivative that better fits our purposes.

2 Study via Fourier Transforms

In fact we very much accomplish that. So utilizing research on Fourier Analysis, we use a definition of the Fourier Transform:

$$\hat{f}(k) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$$

The definition of the inverse Fourier Transform:

$$f(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k) dk$$

. In examining how these definitions operate, we see that the Fourier Transform acts as a basis change from X-basis to K-basis. The K operator is directly related to the Derivative operator D by a coefficient of i. This is shown in the mathematical introduction of R. Shankar's "Principles of Quantum Mechanics." So effectively, the Fourier Transform acts as a basis change from X to D-basis. Here the Derivative is algebraic, and its order in this basis is the equivalent of raising k to a power. From there we utilize the inverse Fourier Transform to return it to x-basis, where the fuction has thus been derived with an order of our chosen power. This Fourier-Fractional Derivative can thus be expressed mathematically as:

$$\frac{d^{\alpha} f(x)}{dx^{\alpha}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixk} (ik)^{\alpha} \hat{f}(k) dk. \tag{1}$$

So this is the first definition we reached. However, it is important to note that work very similar to this has already been completed. After deriving this a very similar definition was found from Petr Zavada's "Operator of Fractional Derivative in the Complex Plane":

For a function $f(x)$, we have its Fourier Transform $\tilde{f}(k)$. To obtain their formal definition they use two equations:

$$f^{\alpha}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (-ik)^{\alpha} \tilde{f}(k) \exp(-ikx) dk, \alpha > -1$$

and

$$D^{\alpha}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (-ik)^{\alpha} \exp(-ik\omega) dk$$

. These come together for their formal definition:

$$f^{\alpha}(x) = D^{\alpha} f = \int_{-\infty}^{+\infty} D^{\alpha}(x-y) f(y) dy \qquad [8]$$

As you can see these definitions are practically identical. Thus, the work so far has essentially been a recreation of another group's work. So this defines the new starting point of this research. Now we will take this definition and apply it to the sin and cosine functions. The reason for this is that there are many functions in which we cannot easily solve the integral that defines their Fourier transform. Sine and Cosine functions have a very simple Fourier Transform, so they are immediately easy to look at. However, definitions for the Fourier-Fractional Derivatives of these Functions have a great amount of utility. Having definitions for the Fourier-Fractional Derivatives of $\sin(\omega x)$ and $\cos(\omega x)$ in terms of x allow us to operate on any function expanded into a Fourier Series, as the Fourier Series expansion writes a function as a sum of multiple sine and cosine functions. This is especially meaningful since our interest in this topic comes from examining the mathematics used in Quantum Mechanics. In Quantum Mechanics the majority of functions that are dealt with are bounded or repeating functions, which are functions that can be expanded into a Fourier Series. Thus our goal from here is to observe the Fourier-Fractional Derivatives of Sine and Cosine functions in order to define a general definition for any real bounded or repeating function expanded into a Fourier Series.

We will first show an in-depth derivation of the Fourier Fractional Derivative of the sine function, and then a brief derivation for cosine, as the work is very similar. To start we note the the Fourier Transform of $\sin(\omega x)$ is:

$$F(sin(\omega x)) = -i\sqrt{\frac{\pi}{2}}(\delta(k-\omega) - \delta(k+\omega)).$$

We will use our definition to solve for orders of $\frac{1}{n}$, $\frac{2}{n}$, and search for a possible pattern. Thus we will assume that this pattern holds for an order of $\frac{m}{n}$ and show that it holds for $\frac{m+1}{n}$ to have a complete induction proof for the definition of the Fourier Fractional Derivative of $\sin(\omega x)$. The derivation is as follows:

$$\frac{d^{\frac{1}{n}}}{dx^{\frac{1}{n}}} sin\omega x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixk} (ik)^{\frac{1}{n}} [-i\sqrt{\frac{\pi}{2}}(\delta(k-\omega) - \delta(k+\omega)] dk =$$

$$\frac{i^{\frac{1}{n}} \omega^{\frac{1}{n}}}{2i} (e^{i\omega x} - (-1)^{\frac{1}{n}} e^{-i\omega x})$$

$$\frac{d^{\frac{2}{n}}}{dx^{\frac{2}{n}}} sin(\omega x) = \frac{d^{\frac{1}{n}}}{dx^{\frac{1}{n}}} \frac{d^{\frac{1}{n}}}{dx^{\frac{1}{n}}} sin(\omega x) =$$

$$\frac{i^{\frac{1}{n}} \omega^{\frac{1}{n}}}{2i} \int_{-\infty}^{\infty} e^{ixk} (ik)^{\frac{1}{n}} [-i\sqrt{\frac{\pi}{2}} \delta(k-\omega) - (-1)^{\frac{1}{n}} \delta(k+\omega)] dk = \frac{i^{\frac{2}{n}} \omega^{\frac{2}{n}}}{2i} (e^{i\omega x} - (-1)^{\frac{2}{n}} e^{-i\omega x})$$

We then assume:

$$\frac{d^{\frac{m}{n}}}{dx^{\frac{m}{n}}} = \frac{i^{\frac{m}{n}} \omega^{\frac{m}{n}}}{2i} (e^{i\omega x} - (-1)^{\frac{m}{n}} e^{-i\omega x})$$

Now we we will show that this pattern holds for $\frac{m+1}{n}$:

$$\frac{d^{\frac{m+1}{n}}}{dx^{\frac{m+1}{n}}} sin(\omega x) = \frac{d^{\frac{1}{n}}}{dx^{\frac{1}{n}}} \frac{d^{\frac{m}{n}}}{dx^{\frac{m}{n}}} sin(\omega x) =$$

$$\frac{i^{\frac{m}{n}}\omega^{\frac{m}{n}}}{2i}\int_{-\infty}^{\infty} e^{ixk}(ik)^{\frac{1}{n}}[-i\sqrt{\frac{\pi}{2}}\delta(k-\omega)-(-1)^{\frac{m}{n}}\delta(k+\omega)]dk = \frac{i^{\frac{m+1}{n}}\omega^{\frac{m+1}{n}}}{2i}(e^{i\omega x}-(-1)^{\frac{m+1}{n}}e^{-i\omega x})$$

We thus have that it holds that for an order of $\frac{m}{n}$:

$$\frac{d^{\frac{m}{n}}}{dx^{\frac{m}{n}}} = \frac{i^{\frac{m}{n}}\omega^{\frac{m}{n}}}{2i}(e^{i\omega x}-(-1)^{\frac{m}{n}}e^{-i\omega x})$$

.

It is clear that if $m = n$, then $\frac{d^{\frac{m}{n}}}{dx^{\frac{m}{n}}}\sin(\omega x) = \omega\cos(\omega x)$, that is, $(\frac{d^{\frac{1}{n}}}{dx^{\frac{1}{n}}})^n\sin(\omega x) = \frac{d\sin(\omega x)}{dx}$. It is also true for $\cos(\omega x)$. Hence it is true for the Fourier Series expansion of a function.

The derivation for $\cos(\omega x)$ is done similarly. With the Fourier Transform of $\cos(\omega x)$ being:

$$F(cos(\omega x)) = \sqrt{\frac{\pi}{2}}(\delta(k-\omega)-\delta(k+\omega))$$

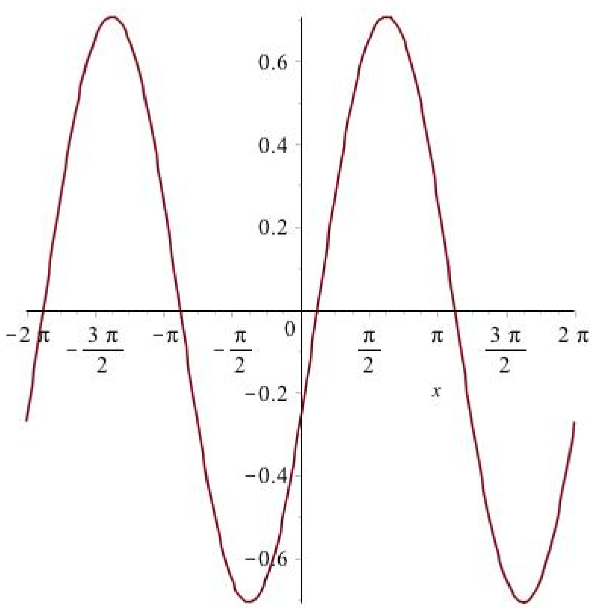
We can derive:

$$\frac{d^{\frac{m}{n}}}{dx^{\frac{m}{n}}}cos(\omega x) = \frac{i^{\frac{m}{n}}\omega^{\frac{m}{n}}}{2}(e^{i\omega x}+(-1)^{\frac{m}{n}}e^{-i\omega x})$$

This gives us the tools we need for our general definition: However, it is first interesting to note some behaviors of this Fourier-Fractional Derivative operator acting on the sine and cosine functions. Noting that every integer derivative of the sine function is a translation of the sine function, one could very well guess that a fractional definition would also be a translation. However, this is just one aspect of our definition of a Fourier-Fractional derivative. This operator is in part a translation Operator when acting on sine and cosine functions, but it also acts as a rotation operator, where it rotates these functions through the complex plane. For an integer n, a derivative of order n is completely real, a derivative of order $n + \frac{1}{2}$ is completely imaginary, and any other order has both real and imaginary parts.

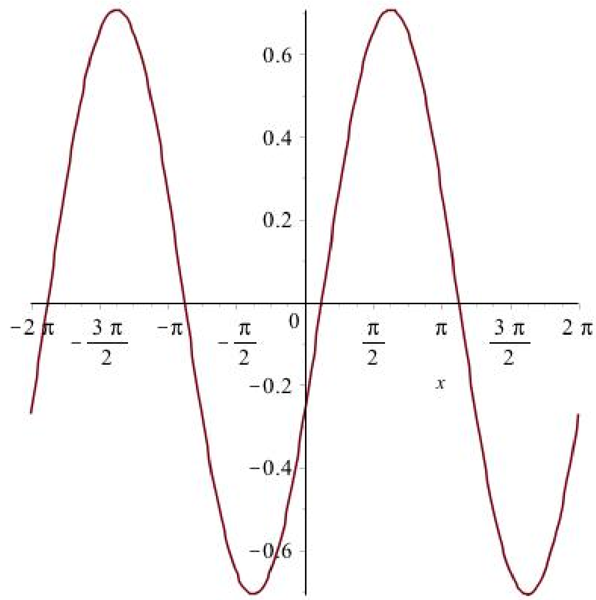
When graphing the real and complex parts of $\sin(x)$ we see for the $\frac{1}{4}$ derivative:

The Real Part:



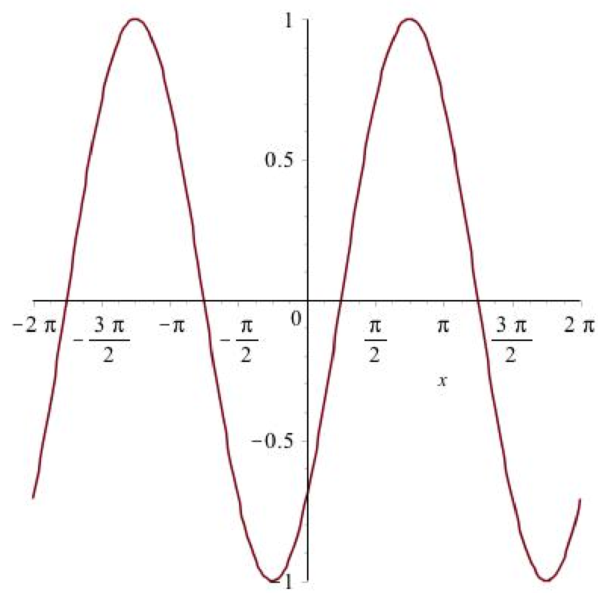
$1_4 Sin.jpg$

The Imaginary Part:



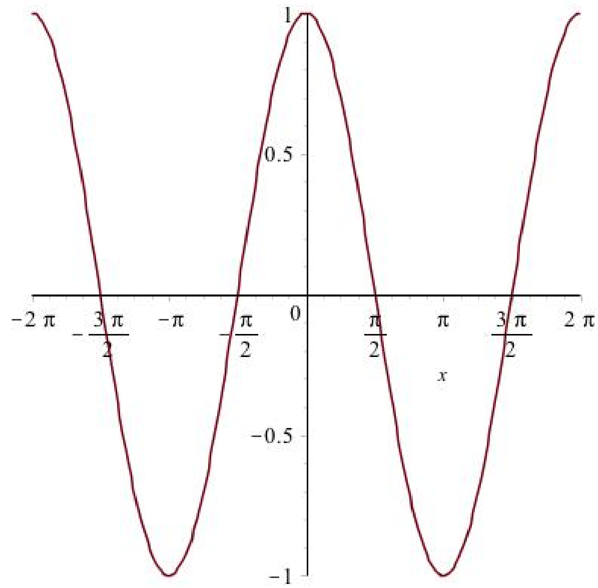
$1_4 Sin.jpg$

For the purely imaginary $\frac{1}{2}$ derivative we will just show its imaginary part:



$2_4 sin.jpg$.

Also we see that for the $\frac{4}{4}$ derivative of sine, the output is cosine, as one would imagine. Since this entirely real we will show the real part:



$sin = cos.jpg$

As is seen by this definition, it holds that the first order derivative of the sine function is the cosine function. It also holds that the first order derivative of the cosine function is the negative sine function. Furthermore, the definition corresponds to the classic derivative definition for every integer-order derivation, and even acts as an anti-derivative operator for negative-integer-order operations. So with these definitions, we will now approach the Fourier Series. We have that the definition of the Fourier Series expansion is:

$$f(x) = \sum_{n=0}^{\infty} a_n cos(nx) + b_n sin(nx).$$

Here a_k and b_k are constants depending on k and on $f(x)$. Now our general definition comes easily:

$$D^{\alpha} f(x) =$$

$$\sum_{n=0}^{\infty} a_n \frac{i^{\alpha} n^{\alpha}}{2}(e^{inx} + (-1)^{\alpha} e^{-inx}) + b_n \frac{i^{\alpha} n^{\alpha}}{2i}(e^{inx} - (-1)^{\alpha} e^{-inx})$$

$$= \sum_{n=0}^{\infty} a_n (D^{\alpha} cos(nx)) + b_n (D^{\alpha} sin(nx)$$

This gives us a definition of a rational ordered derivative for any real, bounded or repeating function. Our goal however is for a Real ordered definition. Thus we need to show a well defined definition for irrational numbers.

Here we now extend our definition to irrational numbers. Let q be an irrational number. Thus there exists a series $\{r_j\}$ such that $q = \lim_{j \to \infty} r_j$ Let $f(x)$ be a function. We have that

$$D^q f(x) = \lim_{j \to \infty} D^{r_j} f(x) =$$

$$\lim_{j \to \infty} \sum_{n=0}^{\infty} a_n \frac{i^{r_j} n^{r_j}}{2}(e^{inx} + (-1)^{r_j} e^{-inx}) + b_n \frac{i^{r_j} n^{r_j}}{2i}(e^{inx} - (-1)^{r_j} e^{-inx}))$$

This is equivalent to:

$$\sum_{n=0}^{\infty} a_n \frac{i^{\lim_{j\to\infty} r_j} n^{\lim_{j\to\infty} r_j}}{2}(e^{inx} + (-1)^{\lim_{j\to\infty} r_j} e^{-inx}) +$$

$$b_n \frac{i^{\lim_{j\to\infty} r_j} n^{\lim_{j\to\infty} r_j}}{2i}(e^{inx} - (-1)^{\lim_{j\to\infty} r_j} e^{-inx}))$$

$$\sum_{n=0}^{\infty} a_n \frac{i^q n^q}{2}(e^{inx} + (-1)^q e^{-inx}) + b_n \frac{i^q n^q}{2i}(e^{inx} - (-1)^q e^{-inx}))$$

Hence we have a definition for an irrational order derivative. Furthermore, since the form in conserved we have a fully defined real-order derivative operator.

With a definition for a real-ordered derivative in place, we can now address the concept of function order derivatives. Let $f(x), g(t)$ be real functions thus we have a definition where $y = \frac{d^{g(t)}}{dx^{g(t)}} f(x)$. By our definition $y \in \mathbb{C}$, and $y(x,t)$ thus we have a four-dimensional system consisting of $x, t, Re(y)$, and $Im(y)$. To simplify this we take the cut $x = t$ in the x-t plane, and use this to plot functions $Re(y)$ and $Im(y)$ as functions of x.

$$D^{g(x)} f(x) =$$

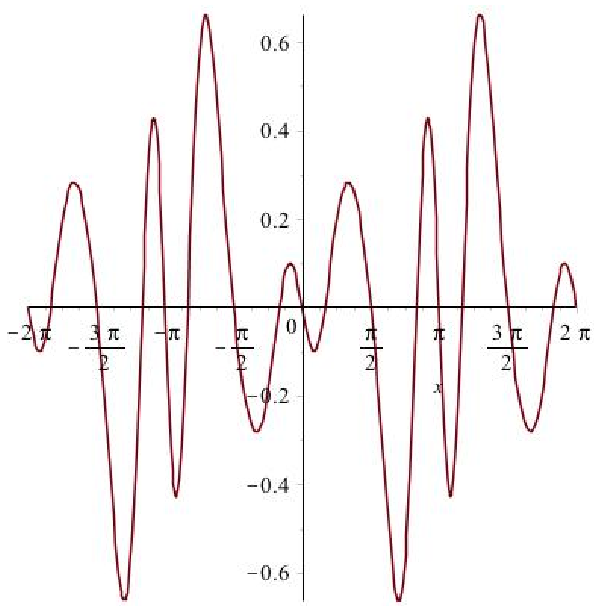
$$\sum_{n=0}^{\infty} a_n \frac{i^{g(x)} n^{g(x)}}{2}(e^{inx} + (-1)^{g(x)} e^{-inx}) +$$

$$b_n \frac{i^{g(x)} n^{g(x)}}{2i}(e^{inx} - (-1)^{g(x)} e^{-inx}).$$

Taking this definition, we arrive at definition that gives readily graphable functions for any real function ordered derivative. Using this we will show some simple $f(x)$ order derivatives for the sine function.

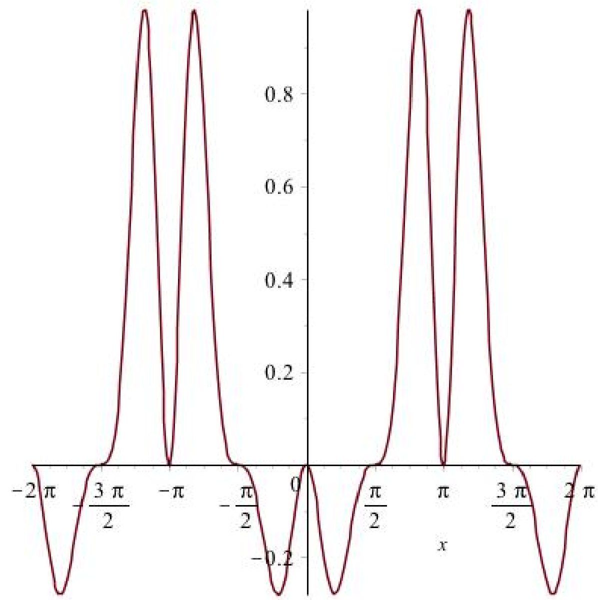
For $f(x) = \sin(x)$:

Real Part:



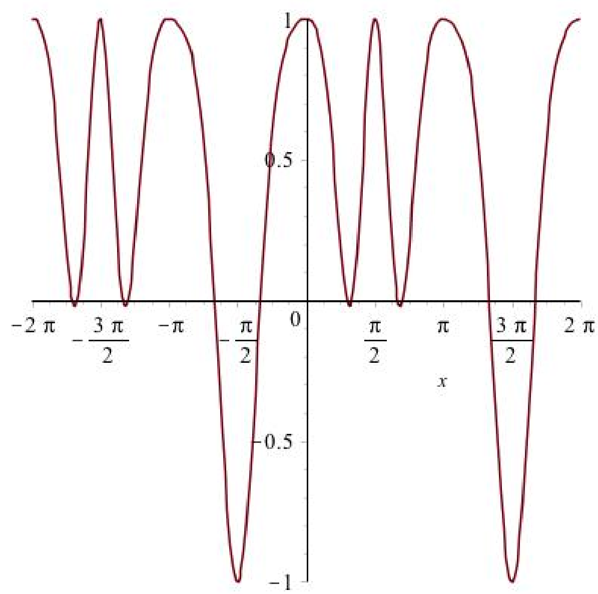
sin(x)sin.jpg

Imaginary Part:



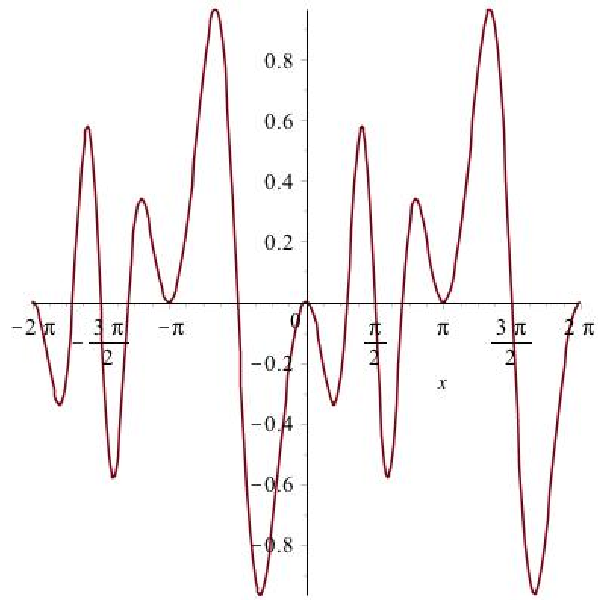
sin(x)sin.jpg

For $f(x) = \cos(x)$:
Real Part:



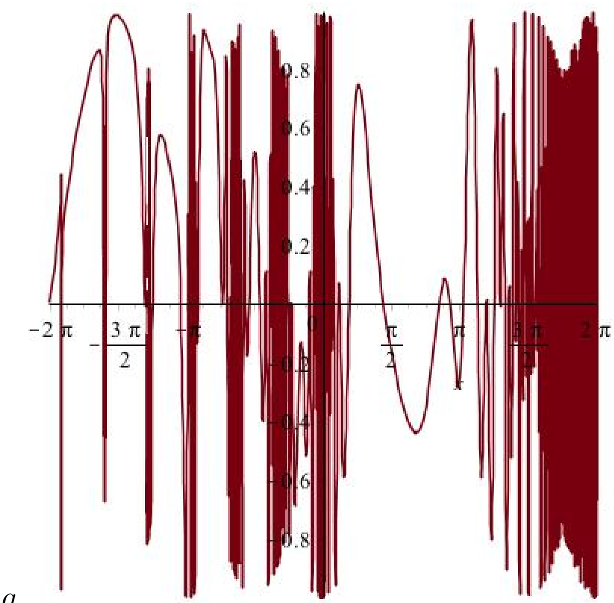
cos(x)sin.jpg

Imaginary Part:



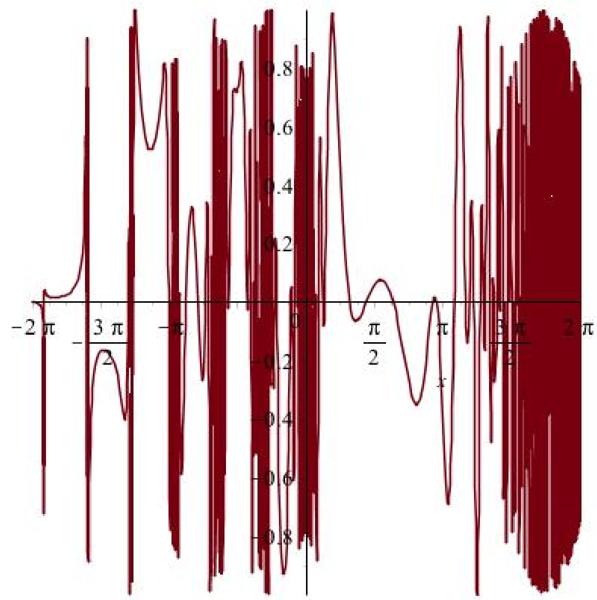
cos(x)*sin.jpg*

Finally for $F(x) = \Gamma(x)$:
Real Part:



gammasin.jpg

Imaginary Part:



gammasin.jpg

3 Open Questions

The attainment of our Function-Ordered Derivative definition and the above graphs, are the pinnacle of this research thus far. Reaching this definition though now acts as a brand new starting point for where this research will go next. There are many properties to explore with this definition and is necessary to find which classical properties still hold and which new properties may have arisen. Particularly, it is important to see if the chain rule, product rule, and quotient rule still hold for this definition. In examining our graphs of the sine and cosine order derivatives of sine, it is interesting to explore the possibility of a relationship between the periodicity of the order, the periodicity of the function being deriving, and the periodicity of the respective derivative. A useful tool to develop alongside this would be to develop an anti-rotating operator, that when applied to our Derivative operator, its function is to reverse the rotation through the complex plane, and provide a completely real result. Finally the immediate next step is to now solve and graph the derivatives of functions expanded into Fourier Series and use this to better understand its properties. It would be important to show that on some interval the first order derivative as defined by the Fourier Fractional Derivative of a function expanded into a Fourier Series is equivalent to, or at least a good approximation of, the Fourier Series of the classically defined first order derivative of the same function. This derivative definition provides us with an entryway into a new section of mathematics to explore, and there is a lot more work to be done on the subject.

Works Cited

References

[1] Loverro, Adam. *Fractional Calculus: History, Definitions and Applications for the Engineer.*, Department of Aerospace and Mechanical Engineering, University of Notre Dame. 8 May 2004. Web. 13 July 2016.

[2] Hunter, John and Bruno Nachtergaele. *Applied Analysis.*, ufcdavis. University of California-Davis. 21 July 2005. Web. 13 July 2016.

[3] Shankar, Ramamurti. *Principles of Quantum Mechanics.*, New York: Springer Science+Business Media, 2014. Print.

[4] Hunter, John. *Lecture Notes on PDEs.*, ufcdavis. University of California-Davis. Web. 13 July 2016.

[5] Yue, Hong. *Weak Derivatives and Sobelev Spaces.*, Spring 2016. PowerPoint Presentation.

[6] Bhatia, Rajendra. *Fourier Series.*, Washington D.C.: The Mathematical Association of America, 2005. Print.

[7] Duoandikoetxea, Javier. *Fourier Analysis.*, Providence, Rhode Island: American Mathematical Society, 2001. Print.

[8] Zavada Petr. *Operator of Fractional Derivative in the Complex Plane.*, Communications In Mathematical Physics, 1998. Web 13 December 2016.