Reasoning with Fractions as Measures and Rational Expressions

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ABSTRACT

According to the literature, a common issue that students have with fractions is understanding that fractions have a magnitude (Freeman & Jorgensen, 2015;). Students need fraction experiences with contexts involving length models and seeing fractions as measures (Freeman & Jorgensen, 2015). We designed and executed a teaching experiment to evaluate students' current understanding of fractions as measures. We also investigated whether there was a correlation between Preservice Teachers' (PSTs) fractional reasoning and procedural fluency with rational expressions. The participants involved were fourth graders, and elementary and special education PSTs. We administered pre-assessments, and presented the students with tasks that required them to create and interpret length models and number lines. All data was collected, analyzed, and compared to answer questions such as how do students reason about fractions as measures?

STATEMENT OF THE PROBLEM

Over the past four years I have tutored students in multiple grade levels who were studying various math subjects. These students were all at different levels of mathematical understanding. Each student struggled with something different, whether it was learning how to solve a two-step equation, how to apply similar triangle theorems, or comprehending the unit circle. Along with tutoring, I have observed elementary, middle and college level classes where students also struggled. One of the topics that continually plague students is fractions. Van de Walle, Karp and Bay-Williams (2015) state that it is important for students to understand fractions in order to be successful in later math classes such as algebra, pre-calculus, and calculus.

National assessments continually reveal that students lack a conceptual understanding and procedural fluency in fractions (Van de Walle et. al., 2015). Students' weak understanding of fractions translates into other areas of mathematics and continues to haunt them throughout all of their mathematical endeavors. Because of this we need to teach fractions in a way that builds conceptual understanding, helps students make connections, and helps them see fractions as interesting, important, and applicable (Van de Walle et. al., 2015). Research suggests that some of the reasons why students struggle with fractions are

- "Students think that the numerator and denominator are separate values and have difficulty seeing them as a single value." (Cramer & Whitney, 2010)
- "Students do not understand that ⅔ means two equal-sized parts (although not necessarily equal-shaped objects)." (Van de Walle et. al., 2015)
- "Students have an inadequate conceptual grounding of unit fractions." (Tzur, R. & Hunt, J., 2015)

- Students are taught fraction rules without sufficient background knowledge or reason. Van de Walle et. al. (2015) states, “Teaching such rules without providing the reason may lead students to overgeneralize.”

It would be most beneficial to students if we could take the time to focus on and clarify their misconceptions; however we cannot focus on all of the misconceptions at once. For this study we would like to focus on students’ understandings of fractions as measures, “that is, understanding both the relative size of fractions (e.g., 3/4 is a bigger number than 1/2) and understanding how fractions measure specific intervals” (Freeman, D. W. & Jorgensen, T. A., 2015, p. 414). There are several researchers who claim that studying fractions as measures help students “understand fractions as a number and helps develop other fraction concepts.” (Van De Walle et. al., 2015). Therefore we will observe students as they reason with fractions when they are presented as measures. The objective of our study is directed to answering the following questions:

1. How do students understand fractions as measures?
2. What misconceptions do they currently have?
3. Are the misconceptions and partial understandings that elementary students have concerning fractions still a hindrance for college students, specifically those who are preparing to be teachers?
4. Do college students use their knowledge of fractions as measures in rational expression tasks?

LITERATURE REVIEW

Fractions as Measures

Freeman and Jorgensen (2015) discuss how more often than not fractions are presented in a part whole model with contexts involving cookies, brownies, and pizza. While these are effective models, they are limited. Freeman (2015) designed a teaching experiment for his fourth grade class that would take place over a period of five weeks. Freeman and Jorgensen (2015) state, “Students need opportunities to develop and explore their understanding of fractions as measures.” Fractions as measures, as previously seen, is defined as “understanding both the relative size of fractions (e.g., 3/4 is a bigger number than 1/2) and understanding how fractions measure specific intervals” (p. 414). The three distinct goals of the experiment were:

1. “Build students understanding of fractions as numbers with a definite magnitude.
2. Increase students’ understanding of measuring with fractions.
3. Develop fraction number sense by avoiding early introduction to traditional fraction algorithms.” (Freeman & Jorgensen, 2015, p. 414)

Students worked on the tasks they were given and then presented them to each other in an open class discussion. As a result of Freeman’s study students gained a better understanding of fractions, established the beginning ideas that lead to the common denominator approach, used the number line to represent fractions, and increased their number sense, according to the Georgia Standards of Excellence (GSE) is defined as “the ability to think flexibly between a variety of strategies in context” (2019, p. 8). Overall they increased their ability to reason with rational numbers.

Rational Expressions

Students who have success reasoning with rational numbers often have greater success with algebra (Yantz, 2013). However, the literature suggests that students struggle with algebra because of a lack of understanding of rational expressions (Randolph, 2015; Yantz, 2013). According the Georgia Standards of Excellence rational expressions are first introduced in Algebra II courses, and they are present in other advanced mathematics courses (2019). For example, Yantz (2013) reviewed several Pre-calculus textbooks and found that rational expressions are often covered in the initial algebra review section. Considering many of the students we were observing will one day be elementary or special education teachers we thought it would be interesting to see their current level of rational expressions knowledge. Regardless of the students success with rational expressions perhaps it will remind them to establish a sure foundation of rational numbers for their future students.

Yantz (2013) executed a study at a southeastern university analyzing students in pre-calculus and calculus classes to determine their understanding of rational expressions. When analyzing the results of her study she noted that students had a great deficiency when working with rational expressions. Yantz (2013) proposed that the success of students in STEM fields would increase if they had a better understanding of how to operate with rational expressions. This motivated our desire to know if students use their knowledge of fractions as measures when reasoning with rational expressions.

Rational Numbers Project

As mentioned before success when reasoning with rational expressions can be linked to a conceptual understanding of rational numbers. In 1979 Behr, Post and Lesh began The Rational

Numbers Project (RNP) based out of the University of Michigan. The goal of their research was to “investigate student learning and teacher enhancement” (Cramer, 2019). They constructed multiple teaching experiments that addressed how students reasoned with rational numbers in multiple forms such as part whole comparison and fractions as measures. It was also designed to help students become better learners, teachers become better teachers, and to help teachers understand students’ understanding. “An important outcome from the early RNP work includes this deeper understanding of children’s thinking as they develop initial fraction ideas” (Cramer, 2019). Part of their success in achieving these goals is the curriculum they developed. It is because of the success of the RNP that we decided to adapt part of this curriculum for the purposes of our study.

Number Line

One tool that proved useful in multiple lessons developed in the RNP and in Freeman’s classroom was the number line. A number line is a visual representation of numbers marked along intervals on a line or line segment. It is useful for representing real number operations, defining the magnitude of a fraction, and showing relationships between and among real numbers. Number lines help develop students’ fractional understanding by:

1. “developing their sense of magnitude of fractions” and seeing that the numerator and denominators are not two separate values, but one number. (Van de Walle et. Al, 2015, p. 344).
2. “organizing thinking about numbers” (Kilpatrick et al., 2001, p. 87).
3. “expanding their number system beyond whole numbers” (Van de Walle et. Al, 2015, p. 344).

Representations and Connections

While Freeman (2015) suggests that the number line is a logical context for exploring fractions as measures, it is important to use multiple representations for students to gain a valuable, conceptual understanding. For example, a tutoring student I once worked with was presented with a problem similar to the one below.

> *Find the total volume of dirt to be dug for the installation of a rectangular pool if the length is 42 feet, the width is 13 feet, and the depth is 6 feet.*
>
> *When the dirt is taken out of the ground it occupies 25% more space. What will the volume of the container need to be to hold the loose dirt?*

The problem required the student to use multiple parts of her prior knowledge and to build off of that knowledge. To make connections between concepts students have not seen in the past, we need to activate their prior knowledge. She understood the first part of the question, but the second part caused perturbation. It was not enough to use a simple drawing; I had to construct a physical model, using brown sugar and measuring cups, for her to understand what was going on in the problem and what mathematical operations would need to be performed. After observing this model the student understood and was able to make connections to the simple drawing and the mathematical operations. Students tend to understand better when they are able to make connections between multiple representations. The figure below depicts how different ways of demonstrating mathematical understanding are all connected. The goal is to make as many connections as possible between and among mathematical ideas.

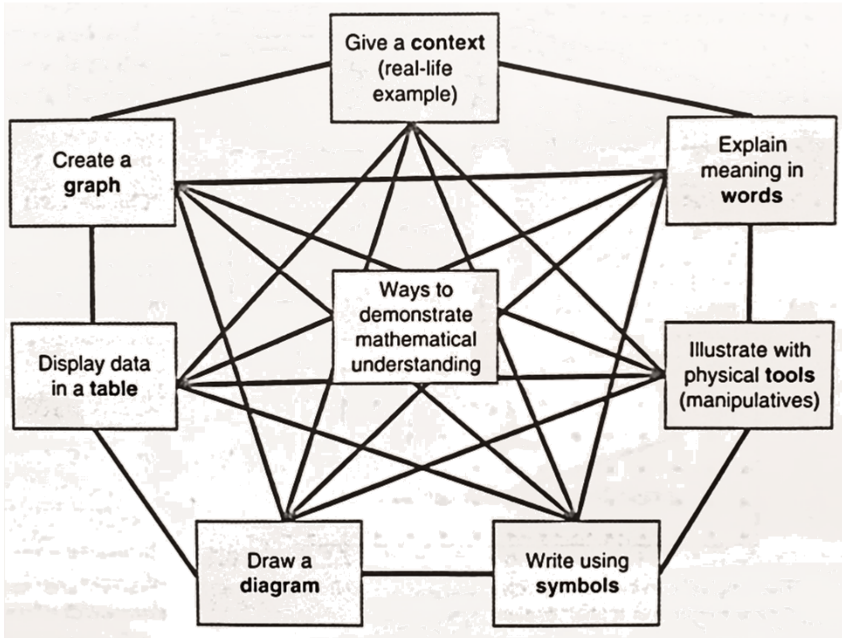


Figure 1. (Van de Walle et. al., 2015, p. 21)

Mathematical Proficiency

Being able to make connections between multiple representations is a sign of mathematical proficiency. According to Kilpatrick, Swafford, Findell (2001) mathematical proficiency consists of five strands that work together (see Figure 2). Conceptual understanding is the "comprehension of mathematical concepts, operations, and relations" (Kilpatrick et al., 2001, p.5). Procedural fluency is the "skill in carrying out procedures flexibly, accurately, efficiently, and appropriately" (Kilpatrick et al., 2001, p.5). Strategic competence is the "ability to formulate, represent, and solve mathematical problems" (Kilpatrick et al., 2001, p.5). Adaptive reasoning is the "capacity for logical thought, reflection, explanation, and justification" (Kilpatrick et al., 2001, p.5). Productive disposition is the

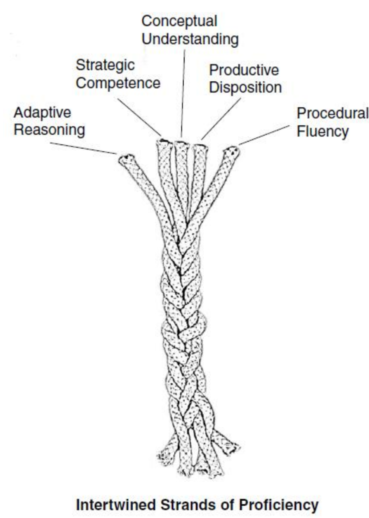


Figure 2: Five Strands of Mathematical Proficiency (Kilpatrick et al., 2001, p.5)

“habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy” (Kilpatrick et al., 2001, p.5). For the purpose of this study I chose to focus on the concepts of conceptual understanding and procedural fluency. That is determining how students currently understand fractions as measures and rational expressions and whether their understanding tends to be more conceptual or more procedural.

METHODS

In order to answer the questions indicated in our problem statement, we designed a teaching experiment using primarily qualitative methods and some quantitative methods. The experiment was performed in a fourth grade classroom and in a college level class of Pre-service Teachers (PSTs), students studying to become elementary and special educators. The tools used for this experiment include a Rational Expression Assessment (REA) and a Fraction Concepts Assessment (FCA). These tools were used to quantify students’ prior knowledge. A detailed lesson plan was used to guide the instruction and sequence of tasks on Fractions as Measures. A Monitoring Tool, Audio Recording Devices, and still cameras were used to document students’ work so that we would be able to recognize and characterize students’ reasoning. The following will describe what happened in each classroom.

College Class

The first day in the Pre-service Teachers’ (PST) classroom we began by having each student complete the Rational Expressions and Fraction Concepts Assessments. While the students were engaged in the assessments, the professor of the course and I observed and answered questions. Immediately following we began Part 1 of the lesson as outlined in the lesson plan (See Appendix A). It is important to note that during the lesson tasks, we were walking around to observe students' strategies and ways of operating. The following prompt began Part 1:

> "You live one mile from Dairy Queen. So you decide you are going to walk to Dairy Queen to get an ice cream cone. Along the way you see the playground, which is $\frac{1}{3}$ mile from your house, and decide to stop and play. After a while you continue walking towards Dairy Queen when you see the library, which is $\frac{3}{4}$ of a mile from your house. Therefore you stop at the library to check out a book."

Students were tasked to partition their paper strip to depict where the playground was located by partitioning it into thirds. The paper strip served as a length model since they were only partitioning in one direction. Next, the students were asked to depict where the library was located by partitioning their paper. After the students completed their task, we lead a group discussion to wrap up the task and clear up misconceptions. Following this task the lesson continued with Part 2:

> "You live two miles from school. As you are walking to school you meet your friend $\frac{2}{5}$ of a mile from your house. After talking to your friend you both continue walking to school when you see your teacher 1 $\frac{1}{5}$ of a mile from your house. You, your friend and the teacher talk while continuing to walk to school."

This time the students were asked to create a length model depicting 2 miles, then choose an appropriate partition that would allow them to mark where $\frac{2}{5}$ and 1 $\frac{1}{5}$ miles were. This concluded the first day of the lesson.

On the second day, students were given their length models and a piece of construction paper and were asked to create a number line based on their length model(s). After they constructed their number line they answered the following questions individually:

1. Please explain why you chose each label on your number line.
2. How are the two pictures alike?
3. How many units are shown in the paper-folding picture?
4. How many units do you see on the number line?
5. Where should we put the numbers 0, 1, and 2 on the number line?
6. Please include any additional comments about your construction.

A group discussion allowed us to determine how they chose to partition, what their misconceptions were, and any other questions or comments they had.

Elementary Class

The fourth graders began with Part 1 and the same tasks that were given to PSTs' class. This concluded the first day. On the second day we began by asking the students to construct a number line based on their length model from Part 1. Then we came together at the end of the task to discuss and clarify the results of the task. After this we extended the task by presenting them with a simplified version of Part 2:

> "You live 2 miles from school. As you are walking to school you stop and talk to your best friend 1 ¼ miles from your house."

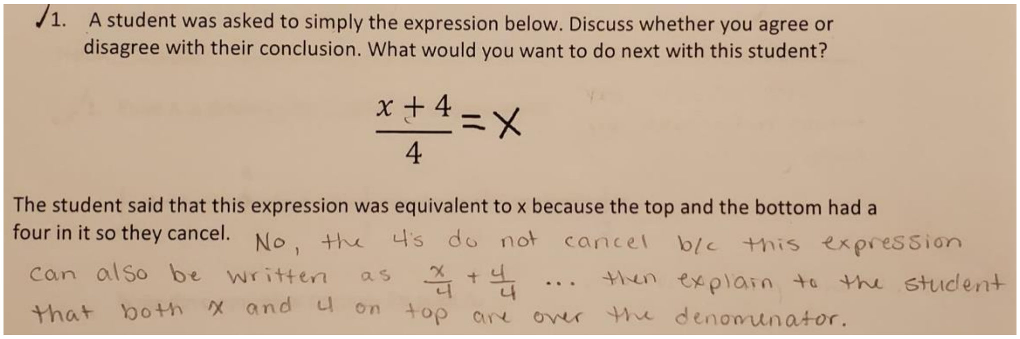
This task was done as a group task. We asked the students to extend the number line on the board, and represent the school and the point where they stopped to talk to their friend. Students responded by approaching the board, consulting with classmates, and deducing the final result.

DATA ANALYSIS/ FINDINGS

Do college students use their knowledge of fractions as measures in rational expression tasks?

The Rational Expressions and Fraction Concepts Assessments were designed to provide a basis of the PST's current understanding of fractions and rational expressions. I had hoped to see students exceeding on the FCA because it covered topics such as fraction operations, equivalent fractions, and the multiplicative identity to find equivalent fractions. These are all elementary and middle grades standards according the Georgia Standards of Excellence (2019). Another intended goal of the FCA was to remind the PSTs of certain fraction concepts to aid them on the REA. The REA covered topics such as rational expressions operations $\left(\frac{x^2+x}{5} * \frac{25}{xy+x}\right)$, simplifying rational expressions $\left(\frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right)$, and finding the domain of a rational expression $\left(f(x) = \frac{x^2-1}{x-1}\right)$.

The first question on the REA presents a very important topic concerning simplifying rational expressions. As indicated in Figures 3 and 4, the problem was already worked by a fictional student. The PSTs were to determine if the work was correct or incorrect and why. The PST class was divided almost exactly in half concerning the accuracy of the fictional students work. Below are examples of the two most common answers, one correct (see Figure 3) and one incorrect. (see Figure 4)

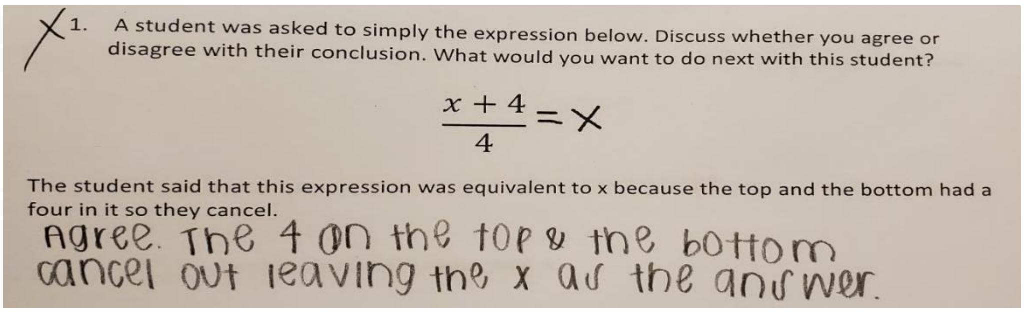


1. A student was asked to simply the expression below. Discuss whether you agree or disagree with their conclusion. What would you want to do next with this student?

$$\frac{x+4}{4} = x$$

The student said that this expression was equivalent to x because the top and the bottom had a four in it so they cancel. No, the 4's do not cancel b/c this expression can also be written as $\frac{x}{4} + \frac{4}{4}$ ... then explain to the student that both x and 4 on top are over the denominator.

Figure 3: Susan's REA Question 1



1. A student was asked to simply the expression below. Discuss whether you agree or disagree with their conclusion. What would you want to do next with this student?

$$\frac{x+4}{4} = x$$

The student said that this expression was equivalent to x because the top and the bottom had a four in it so they cancel.
Agree. The 4 on the top & the bottom cancel out leaving the x as the answer.

Figure 4: Emily's REA Question 1

Each assessment was graded, and the results are shown in the Assessment Comparison Scatter Plot (see Figure 5) below with the FCA grades on the vertical axis, and the REA grades on the horizontal axis. As indicated in the scatter plot there is no correlation between the FCA and the REA. Consequently it is apparent that there is no evidence to suggest that college students use their knowledge of fractions as measures when reasoning with rational expressions.

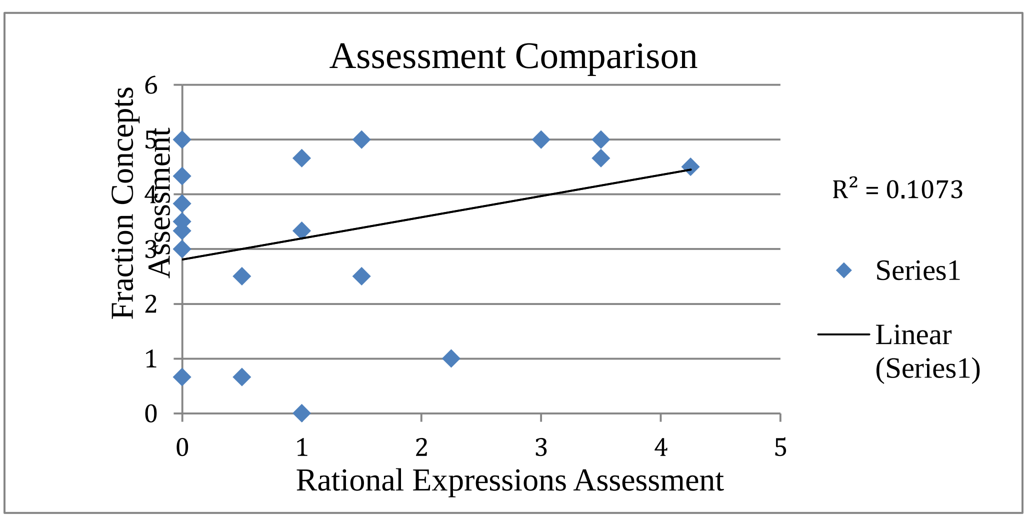


Figure 5: Assessment Comparison Scatter Plot

The Rubric

To analyze the data and answer the research questions we began by looking at each length model and number line construction. As a result we developed the following rubric (see Figure 6) based on our intended goals and the students' reasoning.

Rubric for Analysis		
Folding Techniques	**Labeling**	**Transferring to Number Line**
Seemingly random folds. Using a non-compatible number of folds to partition into the unit fraction 1/b. i.e. Folding into 8 equal parts in order to partition into thirds.	Labels for landmarks are in the spaces rather than the lines or no clearly marked labels at all. For instance, the playground, which was 1/3 of a mile from home, could appear at the ½ mark.	Created a number line with inaccurate, or no, partitions and incorrect labels, or failed to create a number line.
Not ending up with a compatible number of partitions, but then "tearing off" part of the original strip so that a compatible number of partitions emerged. The whole is conflated for the different families (thirds vs. fourths) of fractions, but thirds and fourths are marked by a fold on the different wholes. Only representing one family of fractions and not the other.	Placing the landmarks on a line, but not at the indicated fraction, but rather a convenient, close mark. For instance: placing the playground at the ¼ mark rather than the 1/3 mark because the strip was already folded into fourths. Placing the landmarks in the spaces but then using an arrow to denote the correct line the landmark was supposed to be placed on. Only labeling one family of fractions.	Number line had correct partitions but incorrect labels. Another case may include the child making more than one number line to denote the location of the landmarks. One of the families are relatively even with respect to the whole, but the other family is not.
Strategically determined partitions by folding into "equalish" parts, where each part is the unit fraction, 1/b. Using pencil lines to determine where each part is the unit fraction, 1/b.	Labels for the particular landmarks appear on the appropriate lines rather than the spaces between.	The number line has relatively even appropriate partitions with correct labeling of landmarks for both families of fractions (thirds & fourths).

Figure 6: Rubric for Analysis

How do students reason with fractions as measures? and What misconceptions do they currently have?

No two students constructed the same length model or number line, however many had similar misconceptions. We will use the Rubric for Analysis (see Figure 6) to explicitly define and explain these misconceptions, and students' reasoning with fractions as measures.

Folding Techniques

Figure 7 is an ideal example of student work. We can clearly see that this student understood how to partition their length model, using blue to represent fourths and orange to represent thirds. She partitioned the paper strip into fourths by folding the paper in half then folding each

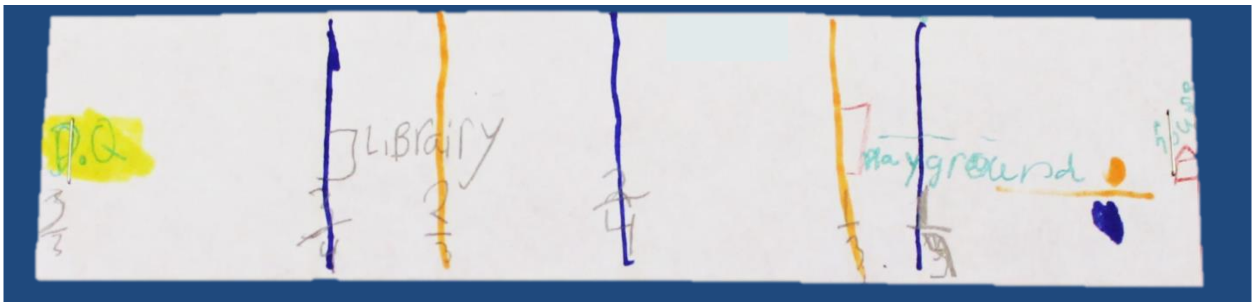


Figure 7: Carmen's (4th) Length Model Part 1

end towards the middle. To partition into thirds she did not fold the paper, she simply drew the orange lines to represent the thirds partitions. While she seemed to strategically place her partitions for thirds, she made no attempt to partition her paper strip into smaller pieces, potentially moving towards the common denominator approach.

Other folding methods for fourths include students using an accordion fold. Students would approximate the first fourth then alternate the remaining folds to create four fourths. Similar to the accordion fold was what I call the roll fold. After approximating the first fourth the students would continue by folding on the same side of the paper strip instead of alternating sides. Another folding method students used was, what I call, the double halves method. They would fold their paper strip in half then in half again, creating fourths. Partitioning into thirds students primarily used two methods, a pamphlet fold, or the roll folding method mentioned before. The pamphlet style fold consisted of the students folding the left side of the paper strip in then the right over the top and adjusting the paper until the thirds were approximately even.

It is interesting to point out that both Carmen (see Figure 7) and Amy (see Figure 8) constructed their length models from right to left. A common convention, which was used by all of the PSTs, says that home, or the starting point, should be on the left most part of the paper strip. However fourth grade students have not been subjected to quite as many conventions just yet. Therefore, I saw the aforementioned length models and several others. These students have not yet conformed to the convention. It would be interesting to go back and ask these students how they decided where to start the point they called "home" on their model?

One of the misconceptions associated with the folding techniques was conflating the whole. Some students when attempting to partition into an odd number of partitions, such as thirds, would partition their paper strip into an even number of folds, such as fourths, to represent that odd number of partitions. The students would then disregard, or tear off, the end space of the paper strip and use what remained of the paper strip to represent a new, smaller whole. I referred

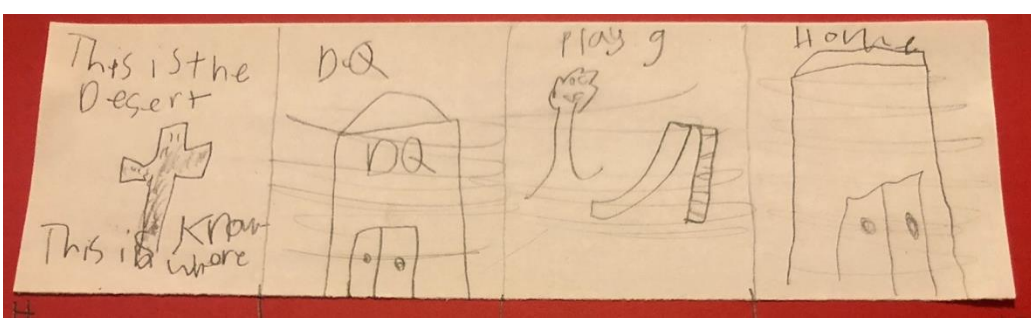


Figure 8: Amy's (4th) Length Model Part 1

to this as the tearing off method. Amy (see Figure 8) disregarded the last piece of her fourths by noting that it was “the desert. This is knowhere.”. She then used the remainder of her paper strip to represent three thirds.

Rachel also used the tearing off method, quite literally. After multiple failed attempts, seen on the second paper strip of Figure 9, she partitioned the strips in to sixths and then tore off

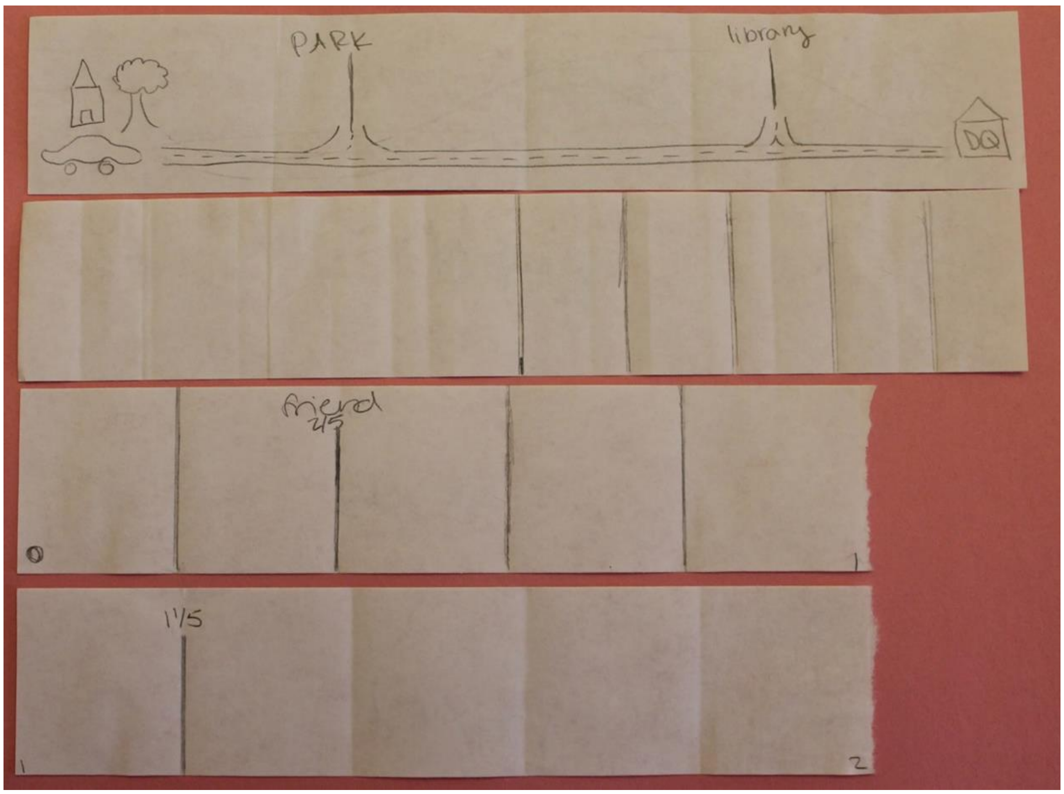


Figure 9: Rachel’s (PST) Length Models Part 1

one sixth so that the new whole would consist of five fifths. As we completed Part 2, transferring to the number line, Rachel successfully completed the remaining tasks. Her number line had relatively even partitions and she correctly labeled the given landmarks.

Another misconception was students partitioning their paper based on a power of two. One PST, when attempting to construct her length model, was insistent that she needed to use 2^3 partitions because she was looking for thirds. After attempting to fold multiple paper strips into eighths she learned it was impossible to find one third when looking at eighths. In other words she could not find an equivalent fraction with a denominator of 8 and a numerator of a whole number. However, a complex fraction, such as $\left(\frac{2\frac{2}{3}}{8}\right)$ could be used.

Another misconception, which was less common, was using an arbitrary number of partitions. For example, Suzy partitioned her length model into eighths which would have been an acceptable partition for depicting the location of the library, ¾, but it would not have been appropriate for finding the playground, 1/3. However when looking at her work (see Figure 10),

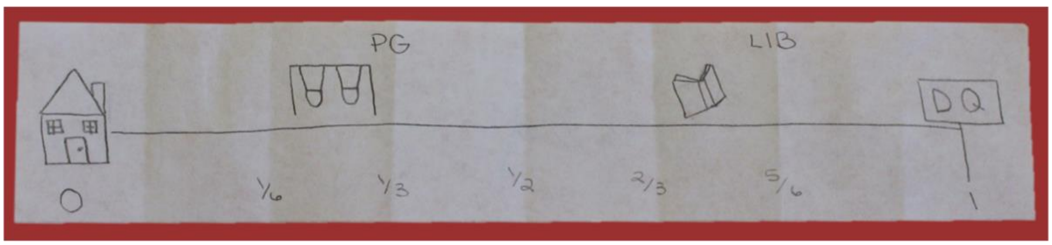


Figure 10: Suzy's (PST) Length Model

it can be seen that she labeled her partitions with the correct ordering but not with a consistent unit.

Labeling

One misconception, which appeared more frequently in the fourth grade class than in the PST class, was labeling for convenience rather than accuracy. For example, Ian (see Figure 11),

Figure 11: Ian's (4^{th}) Length Model

partitioned his paper strip into fourths and properly located the library because he knew how to properly execute that section of the task. However, when it was time to locate the playground one third mile from the home, he did not see one third amongst his partitions. He then chose to label the playground at one fourth mile because it was the closest partition to one third. While he recognized that one third and one fourth were located relatively close to each other, it was surely inaccurate to label the playground at one fourth.

Along with labeling for convenience, labeling in the spaces and not on the partition was another common misconception. This was a mistake that I expected to see, but not in the quantities I observed. Even the PSTs displayed this misconception in their length models. In fact, this was the most common misconception in both classes (see Figure 12). Note that in Halee's

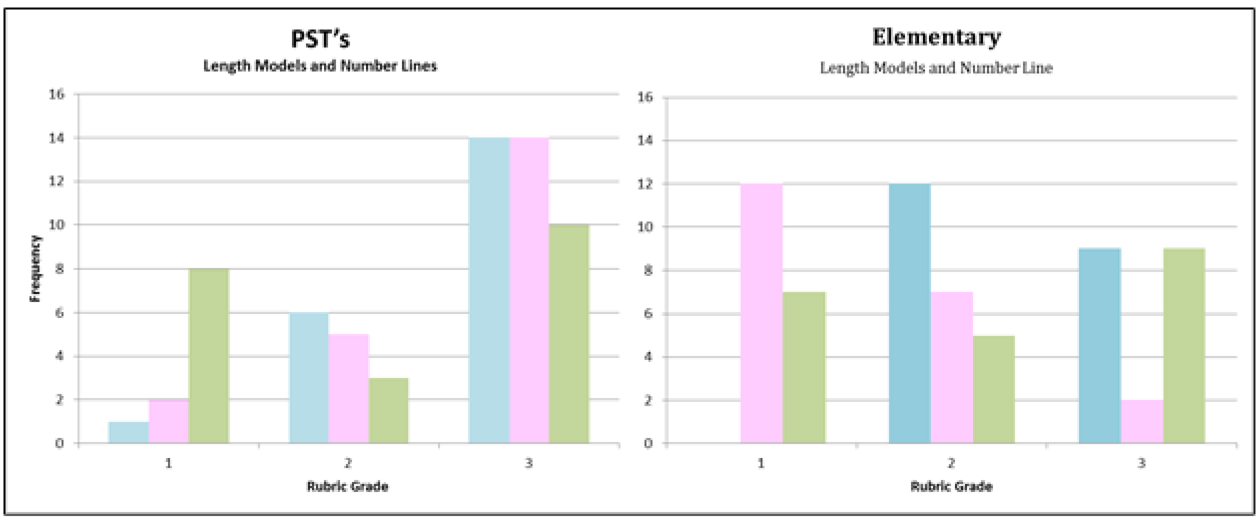


Figure 12: Bar Charts from Rubric for Analysis

length model (see Figure 13) she distinctly labeled each landmark in the spaces. It is interesting to see that she labeled the spaces in both of her length models, considering the class went over the third length model before they were asked to locate the library ¾ mile from the home. After we had a chance to review all of the tasks in Part 1 Halee corrected this error when constructing her length models in Part 2.

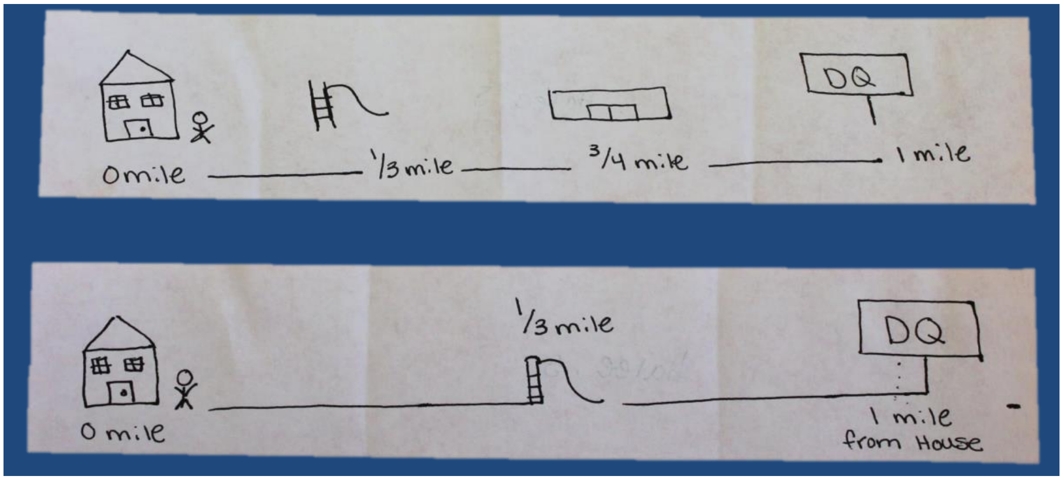


Figure 13: Halee's (PST) Length Models

In the fourth grade classroom we introduced another representation for the class to conceptualize the idea. We took advantage of a rug that was already in their classroom, similar to Figure 14. The rug had a pattern with 30, one by one foot, squares with rows of different

Figure 14: Fourth Grade Carpet

colors. This made it simple for my advisor and I to denote the whole and partitions. My advisor stood on the edge of the carpet, shown by the right most yellow line in Figure 13, and I stood where the leftmost yellow line is in Figure 13, denoting the end of the paper strip, or the whole. My advisor explained to the fourth graders what the prompt said by walking. The students were then asked to tell my advisor to stop when she was "at the playground". The kinesthetic aspect of this representation supports the earlier literature which states that using multiple representations will help students develop connections between mathematical topics. After this physical demonstration some students wanted to act out the prompt for themselves so they, along with the other students in the class who understood by watching, could translate it into their length models. This representation enhanced students' understanding of labeling and partitioning.

Transitioning to the Number Line

When creating a number line based on their length models, I expected students to draw only one number line reinforcing the idea that all real numbers live on one number line. However several students in the fourth grade classroom drew more than one number line to represent their length models. Each line they drew represented a different fraction family because it was easier for the fourth graders to recognize the individual landmarks that way. Jane, a fourth grader, drew (see Figure 15) two different number lines in pencil to represent her length model,

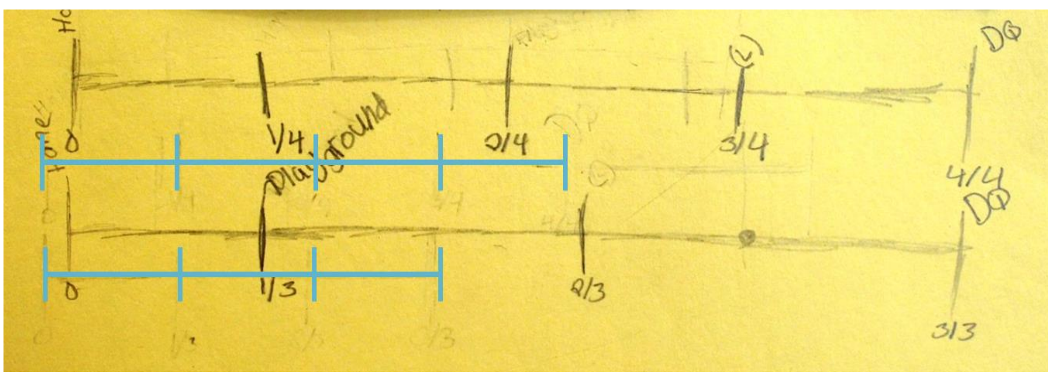


Figure 15: Jane's (4th) Number Lines

which was drawn on one paper strip, not two. We had a class discussion at the end of transferring to the number line, but Jane did not choose to go back and correct her pencil number lines so they were one. Looking closely at Figure 15, highlighted in blue, you can see two number lines of different lengths. These were Jane's original number lines. This was also a case of conflating the whole, previously seen when students were portioning their paper strips. Jane approached me confidently with her original drawing because she had partitioned her number lines correctly, and felt that she had done everything correct. However, if I had said her work was sufficient it may have caused confusion in the future. Therefore my advisor, due to the number of students asking questions, explained how to correct her number lines separately, as seen in pencil. An important aspect of the pencil drawn number lines is that my advisor taught her to take advantage of the length model, paper strip, she made earlier.

Another common mistake made when transitioning to the number line was students having correct partitions, but incorrect labels. This demonstrates the earlier misconception of labeling for convenience (see Figure #). Students would construct their number lines with only tick marks representing both fraction families correctly, but not label one or both fraction

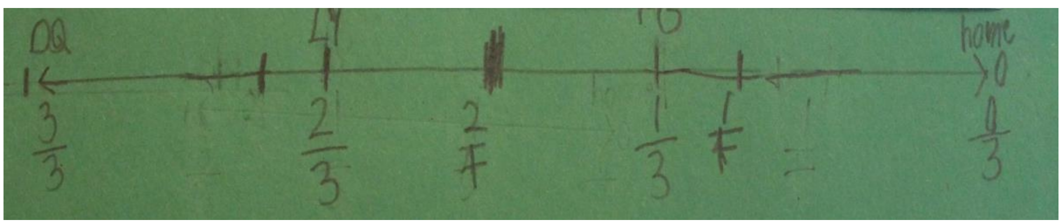


Figure 16: Sheldon's (4th) Number Line

families. After this they would attempt to label the indicated landmarks and put them in the wrong places. Sheldon's work in Figure 16 shows that he only denoted the thirds fraction family and not the fourths; consequently he labeled the library at 2/3 mile. Another, unusual, case was almost the converse of the previous misconception. That is the students would label the landmarks correctly, but with inaccurate partitions. For example, Styles constructed this number

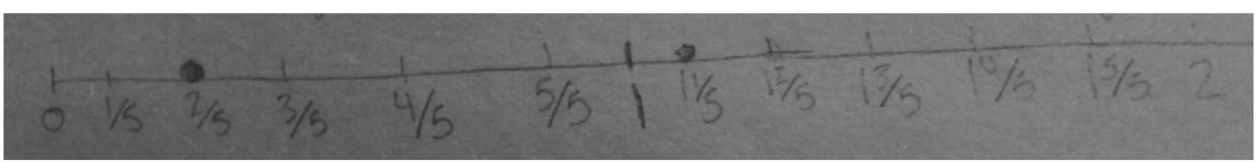


Figure 17: Styles' (PST) Number Line

line (see Figure 17). Initially I was impressed with her choice of partitioning she labeled using mixed numbers, which was an uncommonly used idea in either class. It wasn't until further analysis that I noticed she labeled her number 0, ⅕, ⅖, ⅗, ⅘, 5/5, 1, etc. It was an unusual misconception that I wish I could investigate further to determine if she would repeat what she did in Figure #, or if it was a one-time misconception.

Are the misconceptions and partial understandings that elementary students have concerning fractions still a hindrance for college students, specifically those who are preparing to be teachers?

To answer this question in brief, yes. I expected the fourth graders and the PSTs to have their own unique sets of misconceptions based on their differences in background. However, I was proven wrong. For example, the most shocking misconception that existed in both classes was the misconception of labeling the spaces rather than the partitions (see Figures 13 and 8, Halee and Amy). This misconception afflicted over half of the fourth graders and approximately fifteen percent of PSTs (see Figure 12). Another misconception that plagued both classes was the inability to partition a whole into an odd number of partitions. It was also intriguing to see that the PSTs were perturbed when asked to partition into fifths, or other odd number of partitions. In this class for teachers there were few who were able to fold into fifths on the first try. It took at least two attempts for them to be satisfied with their length models. Overall, it is shown that there are misconceptions that exist amongst the fourth graders that are still apparent for PSTs.

IMPLICATIONS

Amongst the data we collected and the results of analyzing the data there are several implications that became apparent because of this research. From the PST's class I learned that there was no correlation between their reasoning with fractions and their work with Rational Expressions. This is evident because there was no evidence of the PSTs making use of procedures they seemed to understand on the FCA on their REA. As I analyzed the data I also realized that a majority of the PSTs were using rules they remembered from past math classes to complete the tasks and not conceptual knowledge. This made the REA difficult for the PSTs to complete.

One day these PSTs will teach elementary and special education students. I believe that this implies the PSTs need to practice what they are going to teach. Since they are being called to teach a variety of methods and get students to understand conceptually and not just procedurally, they should practice relearning these ideas conceptually. An important note is that the course the PSTs are currently enrolled in is designed to develop the PSTs conceptual understanding of mathematics that they are expected to teach and beyond. When this study was executed the topic of fractions had not been covered yet. My hope is that they recognize the struggles they had so that they can improve their future students' experiences.

How do we avoid students having only a procedural understanding of fractions, and develop their conceptual understanding of measuring with fractions?

In the process of completing this study, three primary methods of improving conceptual understanding instead of merely providing procedural knowledge were identified.

Through the process of analyzing the data we learned that the use of multiple representations (see Figure 1) to describe mathematical problems allows educators to maximize learning modalities in students. For example, I had the students create a length model to capitalize on visual models. Folding that length model provided fine motor manipulation. Verbal descriptions and discussions of the problems provided auditory input and opportunity for self-expression on the students' part. In the fourth grade classroom students struggled understanding why the label for the landmarks was supposed to be on the partition, and not in the space. Capitalizing on the floor rug in the classroom provided the students with a kinesthetic model, which improved their understanding of the problem. Consequently the fourth graders who were confused were able to successfully complete the task. The number line also allowed the fourth graders and PSTs to make connections between the length model and number line. This allowed students to progress from fractions as measures to fractions as a number or location on the number line. Therefore, it is important to present students with multiple representations to solidify conceptual knowledge.

By allowing students to make their own length model from a paper strip they were able to develop partitioning strategies. As mentioned in the Folding Techniques section there were multiple strategies for partitioning different fraction families. Strategies such as pamphlet folding, roll folding, the double halves method, and accordion folding. When students construct their own length models, or manipulatives, they are able to better conceptualize mathematical ideas. This also allows teachers access to students' current understanding so that they can build on their students' current ways of knowing.

In a traditional style lesson students would be shown an example of how to do a particular set of problems and then asked to repeat what the teacher did in order to learn how to complete other problems. At best, this type of lesson would lead to procedural knowledge and an instrumental understanding at best. It does not allow for connections between and among a variety of representations. Unlike a traditional classroom lesson, this project allowed students to build their own manipulatives and come to their own conclusions. Also, unlike a traditional lesson, students' understanding improved when the tasks focused on conceptual understanding rather than procedural understanding. "Understanding these concepts moves children along the continuum toward the increasingly abstract uses of fractions" (Freeman and Jorgensen, 2015, p. 420) which they will see throughout the rest of their mathematical career.

CONCLUSION

This study determined that fourth grade students and Pre-service teachers reason in a variety of ways. This shows that the students were either making use of current information and drawing conclusions, or were remembering and applying previously learned mathematical knowledge. Many misconceptions were identified. Common misconceptions were found in both fourth graders and PSTs. After analysis it can be concluded that the misconceptions that the fourth graders exhibited still hindered the PSTs. It is also known that students have difficulty applying prior mathematical knowledge unless prompted by someone else.

The PSTs improved from Part 1 to Part 2 (see Figure #). This shows that they needed some reminders about what was going on. Considering a majority of them had not been in a math class for over a year, the tasks were very well done.

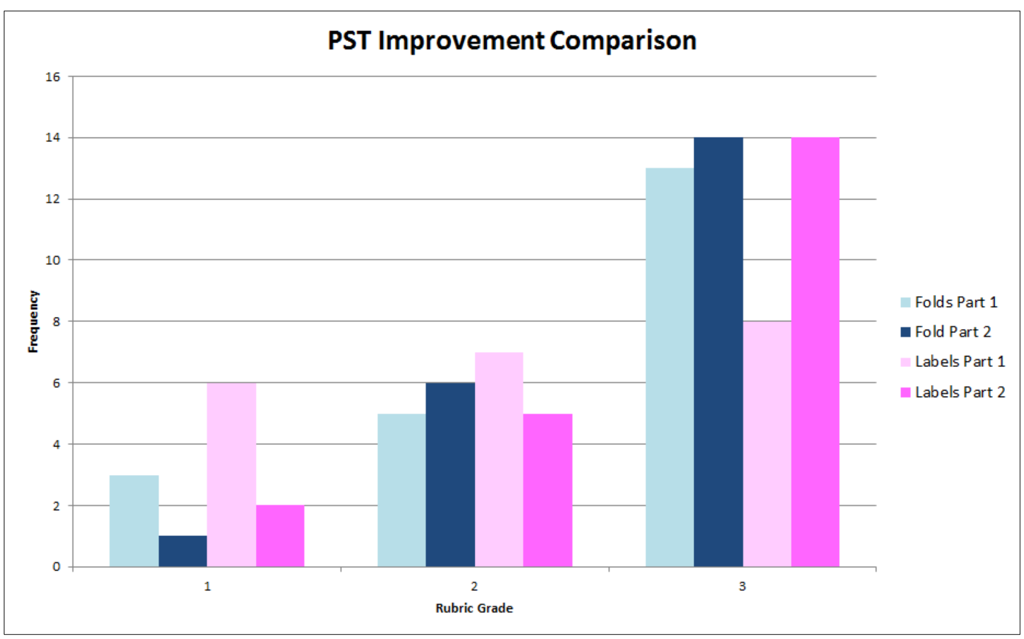


Figure 18: PST Improvement Comparison

When compared to the study performed in "Moving Beyond Brownies and Pizza", their research was executed over a longer period of time and their students were able to recognize approaches, such as the common denominator approach, that they will use later in their mathematical career naturally. My study was executed over two days, for each class. This allowed significantly less time for such realizations. However, the PSTs had prior knowledge of finding equivalent fractions and using common denominators, but did not see it as a relevant tool for locating fraction families involving thirds when fourths were already partitioned. Important outcomes of this research, that are similar to Freeman and Jorgensen's research study, include the students increased their number sense, utilized the number line as a tool, used multiple representations, participated in class discussions, and compared fractions.

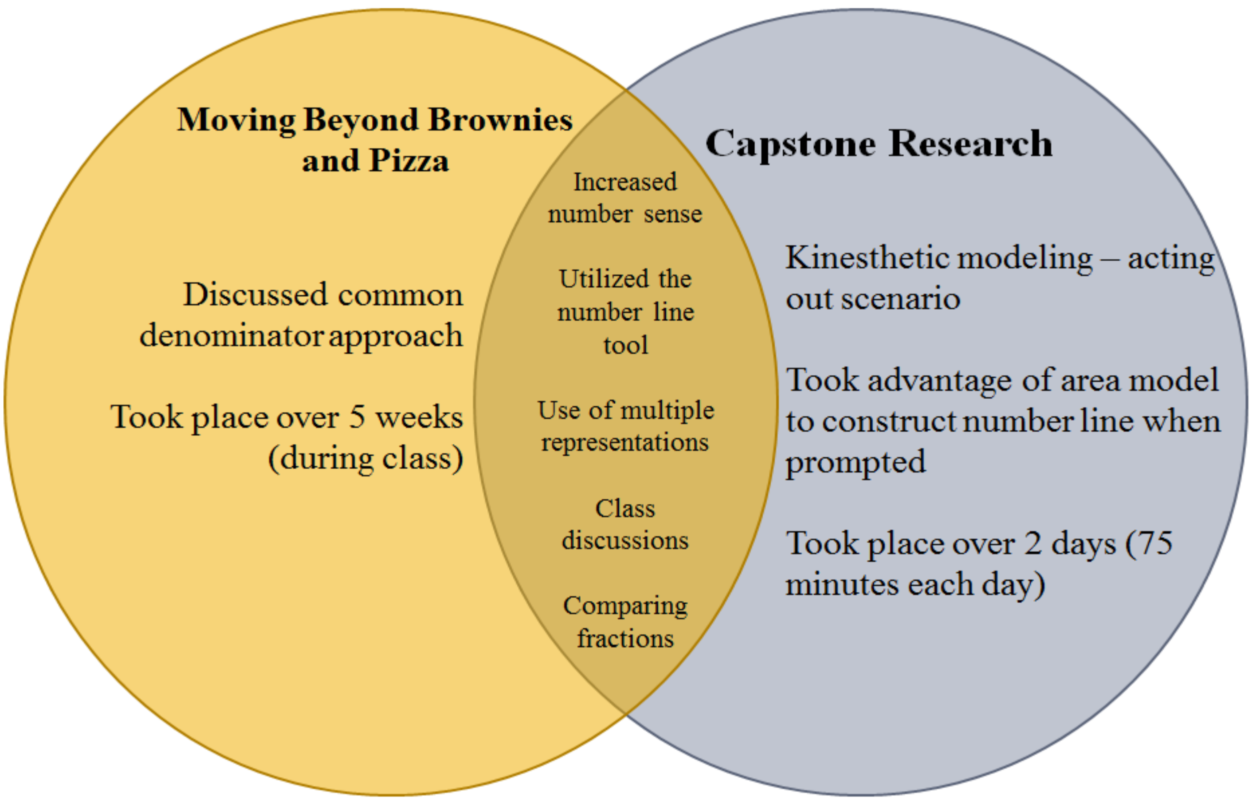


Figure 19: Venn Diagram

The purpose of this research was to observe how students reason with fractions as measures and rational expressions. Before the experiment I tried to plan activities that were more open ended so I could determine how the students were reasoning. I chose the two age groups strategically based on their background mathematics knowledge according to the GSE (2019). While the experiment was being conducted there were productive discussions between tasks that allowed students to communicate with me and with their classmates. Discourse is an essential part of the learning process so that students have the opportunity to reveal, solidify, and extend their understanding. When you learn, or remember something for yourself you are more likely to conceptualize it, even if it takes longer and it causes some perturbation. Though the students struggled, eventually each student did complete all of the tasks. Various partitioning strategies were used and they learned something. In the end the common misconceptions between the two groups cause concern that teachers who only know procedures can only teach procedures. Further studies could address how to increase teachers' conceptual knowledge and educate them to teach from a conceptual point of view.

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APPENDIX A

<table>
<tr><td colspan="3">Fractions as Measures: Addressing a Common Misconception of Fractions</td></tr>
<tr><td>Goal(s): Students will gain a better conceptual understanding of fractions as measures using multiple representations
• Recognize that the resulting interval has size $\frac{a}{b}$ and that its endpoint locates the non-unit fraction $\frac{a}{b}$ on the number line.</td><td colspan="2">Materials:
• - $8\frac{1}{2}'' \times 2''$ white paper strips
• 30 - $8\frac{1}{2}'' \times 11''$ colored paper
• Large paper strips
• Glue
• Markers</td></tr>
<tr><td>Teaching Actions

Warm Up
Order these fractions from smallest to largest:
$\frac{3}{4}$ $1\frac{1}{2}$ $\frac{3}{3}$ $\frac{4}{8}$ $\frac{1}{10}$

Small Group Introduction
1. Present this story: The distance between your home and the Dairy Queen is only 1 mile.
a. Continue the story: While you were walking you wanted to stop at the playground, which is 1/3 miles away from your house. Using one paper strip to represent 1 mile, represent 1/3 of a mile.
b. Continue the story: After playing at the playground you continued walking towards Dairy Queen. Then you remembered you wanted to stop at the library, which was ¾ of a mile from your house. Using a separate paper strip, model ¾ of a mile.

Teaching Moment:</td><td>Questions

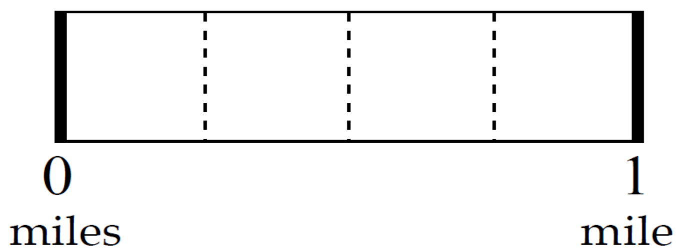
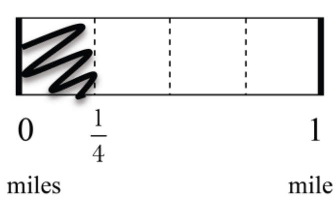
Why does this represent $\frac{1}{3}$ of a mile?

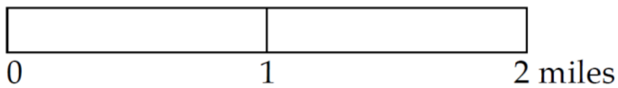
Why does this represent $\frac{3}{4}$ of a mile?

Justify why each fraction strip they folded represents the fraction of the mile.</td><td>Comments/ Time

1(a & b). 10 minutes

2. 15-20 minutes</td></tr>
</table>

2. Using the large paper strip, on the board, show fourths. State the context of the story again: *The distance between your home and the Dairy Queen is only 1 mile.* <u>State that one strip is your unit or whole and stands for one mile.</u> Label the start of the paper strip zero miles.*  	*What does 0 mile represent? - Where should you put the number 1 to show the distance of 1 mile modeled with the paper strip? - How many equal parts is the unit partitioned into? - What is another name for 1 mile based on those partitions? (4/4) - What is another name for 0 miles based on partitions? (0/4)	Partitioned: divided into parts
3. Ask: *You walked ¼ of a mile from your home to the Dairy Queen.* (You may want to act this out by actually walking in front of the picture stopping at the ¼ fold.) Label the fold line at the end of the shaded ¼ part the number ¼. Repeat for 2/4, ¾ and 4/4.  	*Mention equivalent fractions. i.e. 2/4=1/2 - Can you show me where that is on your paper or on the board? - What fraction of a mile is it from 0 to that first partition? - If I shaded 1 of the four equal parts on the paper strip, what fraction of the whole strip is shaded?	**3. 10-15 minutes** Students may label the boxes instead of the fold lines. As we are focusing on the attribute of length and the value of the point at the end of the length, encourage students to label the fold lines and not the boxes.
4. Present this story: *Suppose your school is 2 miles from your house.*		

5. Continue the story: *You walked ¼ of a mile to school.*  6. Continue the story: *You continue walking. After you have walked 1 ¼ miles you run into your teacher.*	- How can we show that using our paper strips? - How many strips are needed to show 2 miles? - Where is ¼ mile in this picture? - How did you get that? How did you decide on your partitions? - Where is 1¼ mile in this picture? - Why is 1 ¼ between 1 and 2?	**4. 5 minutes** **5. 5-7 minutes** **6. 10 minutes**

<table>
<tr><th colspan="3">Fractions as Measures: Addressing a Common Misconception of Fractions</th></tr>
<tr><td>Goal(s):
- Students will understand fractions as numbers with a definite magnitude
- Students will be able to use number lines as a tool to understand fractions as measures</td><td colspan="2">Materials:
• 30 - $8\frac{1}{2}'' \times 11''$ colored paper from Day 1 Lesson
• 30- Number line Student Worksheet
• 30 – Sequence of Fractions Worksheet</td></tr>
<tr><td>Teaching Actions

Small Groups Introduction
1. Remind students of the context of the story begun in the previous lesson: Suppose your school is 2 miles from</td><td>Questions

- Is there anyone who can remind us where the story we were talking about last time ended? –</td><td>Comments/ Time

1. 5 minutes</td></tr>
</table>

your house. You walked ¼ of a mile to school. There you saw your best friend, so you stopped to talk to them. Using your paper strip(s) represent ¼ of a mile. *You continue walking. After you have walked 1 ¼ miles you run into your teacher who reminds you about the fun activity she has planned for class today.* Using your paper strip(s), represent 1 ¼ of a mile.	Refresh story	
2. Draw these two pictures on the board. Ask students to build the number line below their picture of the paper-folding strip on the piece of construction paper.	- How are the two pictures alike? - How many units are shown in the paper-folding picture? - How many units do you see on the number line? - Where should we put the numbers 0, 1 and 2 on the number line?	**2. 10-15 minutes** To emphasize the units, use larger tick marks for 0, 1, 2, etc. Students may put 0 and 2 on the arrows. If they do that help them see that 0 is a number and is a point on the number line. Remind them that the arrows are just a way to communicate that the numbers keep on going (3, 4, 5… or -1, -2, -3…)
3. I see a difference between the number line picture and the picture for paper folding strips.	- What do the arrows at the ends of the line mean?	**3. 3 minutes**
4. Have the students draw a blank number line. Project a blank number line on the Smart Board. Using the numbers from the first day Warm-Up.	- Can you find these fractions on the number line? - Will the partitions be the same for each fraction? - Where does each number lie on the number line?	**4. 10 -20 minutes**

APPENDIX B

1. A student was asked to simply the expression below. Discuss whether you agree or disagree with their conclusion. What would you want to do next with this student?

$$\frac{x+4}{4} = x$$

The student said that this expression was equivalent to x because the top and the bottom had a four in it so they cancel.

2. a) Simplify the following rational function.

$$\frac{x^2-1}{x-1}$$

b) Find the domain for the function, $f(x) = \frac{x^2-1}{x-1}$.

3. Write the rational expression in lowest terms: $\frac{8x^2+16x}{4x^2}$

4. Simplify the expression: $\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$

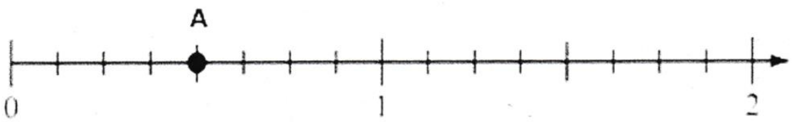
5. Determine the product or sum of the expressions below:

a. $\frac{x^2+x}{5} * \frac{25}{xy+x}$ b. $\frac{1}{x+z} + \frac{1}{x-z}$

APPENDIX C

Name: ____ Date: _________________________

1. Point A is shown on the number line diagram below.



Write three equivalent fractions for point A.

__________________ __________________ ___________________

2. Write a number in every box to make true equations

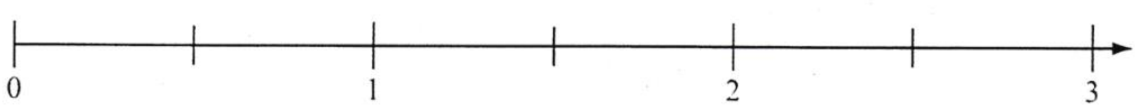
a) $$\frac{7}{5} = \frac{24 \times 7}{\square \times 5}$$

b) $$\frac{15}{10} = \frac{5 \times 3}{\square \times 2}$$

c) $$\frac{67}{100} = \frac{\square \times 67}{\square \times 100}$$

3.

a. Place a point at $\frac{5}{4}$ on the number line diagram below.



b. Write a fraction equivalent to $\frac{5}{4}$. Your fraction must have a denominator of 12. Use words or a diagram to show that your fraction is equivalent to $\frac{5}{4}$.

4. Quan poured $\frac{2}{8}$ gallon of paint into an empty container. Marisa poured $\frac{3}{5}$ gallon of paint into the container. How much paint is in the container now? ____________________ gallon(s).

5. Nicole gives $\frac{6}{8}$ cup of food to each of her rabbits every day. She has 7 rabbits. How many cups of food will Nicole feed to the rabbits every day?

Select the true statement.

a. Between 4 and 5 cups of food every day
b. Between 5 and 6 cups of food every day
c. Between 6 and 7 cups of food every day
d. Between 7 and 8 cups of food every day

APPENDIX D

Monitoring Tool

Strategies	Who & What	Comments & Reflections