Painting polyhedra

An Application of Burnside's Formula

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Abstract

There is an activity that involves figuring out the number of different cubes a person could get if they were coloring the faces of the cube with three different colors. This activity is lengthy and tedious to most, because most people list out all the different possibilities. Is there a way to generalize this activity or change it in any way? The answer to this question can be found by looking at an abstract algebra topic, groups acting on sets. In this research, we will use Burnside's formula and groups acting on sets to find the number of distinct colorings for different shapes. We will vary parameters such as dimension and number of colors.

1 Introduction

We began by looking at the rotational symmetries of a square and a cube. We found that there were four rotational symmetries of a square and twenty-four for a cube. We want to see how we could color these two shapes with different colors. Originally, we looked at the different ways to color the faces of a cube with three different colors up to rotational symmetry. This means that if one coloring can be rotated to get another coloring, those colorings are the same. After creating a paper cube and tediously counting and marking down the different colored cubes we came up with fifty-seven distinct cubes. These are cubes that do not look like any other cubes when rotated. We continued to find the number of distinct colorings for various different shapes, including triangles, tetrahedrons, and hexagons. In the second dimension we color the edges of the shape instead of the faces. We quickly found that trying to visualize each shape was time consuming and became increasingly more difficult with higher dimensions, more colors, and more sides to the shapes. We knew that the different ways of coloring shapes could be generalized by a formula from abstract algebra. This formula is called Burnside's Formula or Burnside's Lemma. In order to understand the formula we will need to define some things first.

2 Mathematical Background

The source of the material in this section unless specified otherwise is Fraleigh. [?]

First, we will define groups acting on sets. A group G acting on a set X is the map $* : G \times X \to X$ such that

a) $ex = x$ for all $x \in X$,

b) $(g_1 g_2)(x) = g_1(g_2 x)$ for all $x \in X$ and all $g_1, g_2 \in G$.

If this occurs then X is a G-set.

Next, we need to know when $gx = x$. Let X be a G-set and let $x \in X$ and $g \in G$.

We denote X_g and G_x in the following way,

$X_g = \{x \in X \mid gx = x\}$ and $G_x = \{g \in G \mid gx = x\}$. X_g is the elements in X such that g takes x to x. G_x is the elements in G that take x to itself.

Next, we will define orbits. If $x \in X$, the cell containing x is the orbit of x. We let this cell be Gx. We can denote Gx as follows, $Gx = \{gx \mid g \in G\}$.

We will also need to know the following lemma.

Lemma 1. *Let H be a subgroup of G. Let $g_1, g_2 \in G$. The following are equivalent.*

a) $g_1 H = g_2 H$

b) $g_1^{-1} g_2 \in H$

We will use this information to help us prove a theorem that is used in the proof of Burnside's Formula, but first, we will prove the above lemma.

Proof. Let H be a subgroup of G. Let $g_1,\ g_2 \in G$. We will show that $g_1H = g_2H$ if and only if $g_1^{-1}g_2 \in H$.

We will start with $g_1H = g_2H$. Let $h_1,\ h_2 \in H$ such that $g_1h_1 = g_2h_2$. We can multiply both sides on the left by g_1^{-1}. Doing so will give us $h_1 = g_1^{-1}g_2h_2$. Then, we will multiple both sides on the right by h_2^{-1}. This yields $h_2^{-1}h_1 = g_1^{-1}g_2$. Since H is a group $g_1^{-1}g_2 \in H$.

Now we will start with $g_1^{-1}g_2 \in H$ and show that $g_1H = g_2H$. By our hypothesis we know that $g_1^{-1}g_2 = h_1$, where $h_1 \in H$. We can multiple on the left by g_1 and get $g_2 = g_1h_1$. Let $g_2h_2 \in g_2H$. We know that $g_2 = g_1h$ so $g_2h_2 = g_1h_1h_2$. H is a group so $h_1h_2 \in H$. Thus, $g_2h_2 \in g_1H$, so $g_2H \subset g_1H$.

If $g_2 \in g_1H$ then $g_2 = g_1h_1$. We can multiple on the right by h_1^{-1} and get $g_2h_1^{-1} = g_1$. Thus, $g_1 \in g_2H$. By a similar process above we get that $g_1H \subset g_2H$. Therefore, $g_1H = g_2H$.

Therefore, if H is a subgroup of G and $g_1,\ g_2 \in G$ then $g_1H = g_2H$ if and only if $g_1^{-1}g_2 \in H$. □

For the following proof we will need Lagrange's Theorem. Recall that $(G : H)$ is the number of left cosets of H in G This theorem states that if H is a subgroup of G, then $|G|/|H| = (G : H)$. Since $(G : H)$ is a whole number then the order of G is divisible by the order of H. The theorem is as follows.

Theorem 1. *Let X be a G-set and let $x \in X$. Then $|Gx| = (G : G_x)$. If $|G|$ is finite, then $|Gx|$ is a divisor of $|G|$.*

Proof. We define a one-to-one map ψ from Gx onto the collection of left cosets of G_x in G. Let $x_1 \in Gx$. Then there exists $g_1 \in G$ such that $g_1x = x_1$. We define $\psi(x_1)$ to be the left coset g_1G_x of G_x. We will show that ψ is well defined regardless of the choice of $g_1 \in G$ such that $g_1x = x_1$. Assume that $g_1'x = x_1$. Then, $g_1x = g_1'x$. We can multiply this equation by g_1^{-1} on the left side. This gives us $x = (g_1^{-1}g_1')x$. Thus, $g_1^{-1}g_1' \in G_x$ by how we defined G_x. By the lemma above we know that $g_1G_x = g_1'G_x$. Thus, the map ψ is well defined.

Next, we will show that ψ is one-to-one. Suppose that x_1,$x_2 \in Gx$ and $\psi(x_1) = \psi(x_2)$. There exist $g_1,\ g_2 \in G$ such that $x_1 = g_1x$ and $x_2 = g_2x$. We also know that because ψ is well defined that

$g_2 \in g_1 G_x$, so $g_2 = g_1 g$ for some $g \in G_x$. Observe the following,

$$\begin{aligned} x_2 &= g_2 x, \\ &= g_1(gx), \\ &= g_1 x, \\ &= x_1. \end{aligned}$$

Thus, ψ is one-to-one.

Lastly, we will show that ψ is onto. Let $g_1 G_x$ be a left coset and let $g_1 x = x_1$. By how we defined ψ, we have $g_1 G_x = \psi(x_1)$. Thus ψ maps Gx one to one and onto the collection of left cosets so $|Gx| = (G : G_x)$.

If $|G|$ is finite, then the equation $|G| = |G_x|(G : G_x)$ shows that $|Gx| = (G : G_x)$ is a divisor of $|G|$. □

Now we have the necessary information to discuss Burnside's Formula.

Theorem 2. *Let G be a finite group and X a finite G-set. If r is the number of orbits in X under G, then*

$$r \cdot |G| = \Sigma_{g \in G} |X_g|.$$

Proof. We will look at all of the pairs (g, x) such that $gx = x$, and we will let N be the number of pairs. For each $g \in G$ there are $|X_g|$ pairs having g as the first member. It follows that,

$$N = \Sigma_{g \in G} |X_g|. \tag{1}$$

We also have that for each $x \in X$ there are $|G_x|$ pairs having x as the second member. It follows that,

$$N = \Sigma_{x \in X} |G_x|.$$

This means that the sum of the orders of all X_g and G_x will be the same. By the theorem that we proved above we have that $|Gx| = (G : G_x)$. We know from Lagrange's theorem that $(G : G_x) = |G|/|G_x|$. Combining the two we get $|G_x| = |G|/|Gx|$. It follows that,

$$N = \Sigma_{x \in X} \frac{|G|}{|Gx|} = |G|(\Sigma_{x \in X} \frac{1}{|Gx|}). \tag{2}$$

$\frac{1}{|Gx|}$ will have the same value for all x that are in the same orbit. We let O be any orbit, so

$$\Sigma_{x \in O} \frac{1}{|Gx|} = \Sigma_{x \in O} \frac{1}{|O|} = 1. \tag{3}$$

If we substitute 3 into 2 then,

$$N = |G|(\text{number of orbits in X under G}) = |G| \cdot r. \tag{4}$$

We can set equation 1 and equation 4 equal to each other to get Burnside's formula. □

We can solve for r in Burnside's formula to get

$$r = \frac{1}{|G|} \cdot \Sigma_{g \in G} |X_g|$$

.

3 Applying Burnside's Formula

Now how does this make finding distinct colorings based on how many colors we have easier? Our group G is the group of rotational symmetries. You can use other symmetry groups such as the total symmetries of a shape. X is the set of all colorings of a shape with n colors. We will use group action with Burnside's Formula. This is when a element of our group G takes and element from the set X to another element in the set. For each $g \in G$, $|X_g|$ is the amount of colored shapes such that when g is applied to the shape it looks the same. We divide the sum of $|X_g|$ by $|G|$. This gives us the number of orbits in X under G. In other words, it gives us the number of distinct colorings that we can have.

Colorings that are taken to other colorings by elements of G are considered the same coloring.

The first shape we looked at with this new formula was the square. We looked at the rotational symmetries of a square (there are four). There is the identity (ρ_0), a 90 degree rotation (ρ_1), a 180 degree rotation (ρ_2) and a 270 degree rotation (ρ_3). We wanted to see how to color the four edges of the square using four different colors. This gives us 4^4 colored squares. This number is not the number of distinct squares, but represents the numbers of squares if we were to label their sides. The identity rotation takes all of these squares to themselves so the order of X_{ρ_0} is 256. The next rotation (ρ_1) takes 4 of the 256 squares to themselves. These are the squares that have all edges colored the same color. ρ_2 takes 16 squares to themselves (squares that have all edges the same color or have opposite edges the same color) and ρ_3 takes 4 squares to themselves (squares that have all edges colored the same color). Thus, the sum of the orders of X_g is $256 + 4 + 16 + 4 = 280$ We then divide this by the number of symmetries we have to get the number of distinct squares we can have with four colors. $\frac{280}{4} = 70$. Thus, there are 70 distinct squares when the edges are colored with four colors.

We observed that $|X_{\rho_2}|$ was four times $|X_{\rho_1}|$ and $|X_{\rho_3}|$. Why is this? It is easy to see if we show the rotations in permutation notation.

ρ_0: (1) (2) (3) (4)

ρ_1: (1234)

ρ_2: (13) (24)

ρ_3: (1432)

If we want to get squares that look the same when we rotate them, then all of the sides in a cycle have to be the same color. There are two cycles for ρ_2 and only one for ρ_1 and ρ_3. Thus, $|X_{\rho_2}|$ is the square of $|X_{\rho_1}|$ and $|X_{\rho_3}|$.

We can generalize the number of distinct squares for n colors. We get the following equation

$$\frac{n^4 + n^2 + 2n}{4}$$

The other 2 dimensional shape that we looked at was the triangle. There are only three rotational symmetries of a triangle. These symmetries are the identity, rotating the triangle 120 degrees and rotating the triangle 240 degrees. We find that the 120 and 240 degree rotations have one cycle. If we had n colors then we can take n and multiply it by the number of cycles for each rotation. This yield the following formula.

$$\frac{n^3 + 2n}{3}$$

The vertices of the triangle will act the same way since each vertex corresponds to a certain edge.

It was not difficult to figure out the $|X_g|$ for the square without using permutation notation because the square does not have many rotational symmetries. However, using permutation notation is extremely helpful when looking at shapes of 3 dimensions and higher.

We already knew that there are 57 distinct cubes when the faces are painted with 3 different colors. We will show this again using the permutation notation of a cube and Burnside's formula. The 24 rotational symmetries of a cube are as follows.

ρ_0: (1) (2) (3) (4) (5) (6)

ρ_1: (1234) (5) (6) ρ_2: (13) (24) (5) (6) ρ_3: (1432) (5) (6)

ρ_4: (1536) (2) (4) ρ_5: (13) (56) (2) (4) ρ_6: (1635) (2),(4)

ρ_7: (5264) (1) (3) ρ_8: (56) (24) (1) (3) ρ_9: (5462) (1) (3)

ρ_{10}: (13) (52) (64) ρ_{11}: (14) (65) (32) ρ_{12}: (15) (24) (36)

ρ_{13}: (13) (54) (62) ρ_{14}: (12) (65) (43) ρ_{15}: (16) (24) (35)

ρ_{16}: (125) (346) ρ_{17}: (521) (643) ρ_{18}: (145) (263)

ρ_{19}: (541) (362) ρ_{20}: (164) (523) ρ_{21}: (461) (325)

ρ_{22}: (162) (543) ρ_{23}: (261) (345)

We are painting the cube with 3 colors, so $|X_{\rho_0}|$ will be 3^6 or 729. $|X_{\rho_1}|$ will be $3^3 = 27$ since there are 3 cycles for ρ_1. There are 11 other rotations that have 3 cycles, so the order of the colored cubes that are the same when these rotations are applied to them is 27. Three of the rotations have 4 cycles, so the order of the colored cubes that are the same when these rotations are applied to them is 81. There are eight rotations that have 2 cycles, so the order of the colored cubes that are the same when these rotations are applied to them is 9. Now we take the sum of all of these orders. $27(12)+81(3)+9(8)+729 = 1368$. Then we divide the sum by the number of symmetries. $\frac{1368}{24} = 57$.

Thus, we see that there are 57 distinct cubes when coloring with 3 colors. This is what we originally found, but by using this process we can easily find the number of distinct cubes for n colors. We used the same process but with 4 colors. The sum of $|X_g|$ is $64(12) + 256(3) + 16(8) + 4096 = 5760$. Then we divide the sum by the number of rotational symmetries $\frac{5760}{24} = 240$. Thus, there are 240 distinct cubes when the faces are painted with 4 colors. We can generalize this to an arbitrary number of colors. The cube has 12 rotational symmetries with 3 cycles, 3 rotational symmetries with 4 cycles and 8 rotational symmetries with 2 cycles. If the amount of colors is n then the 12 rotational symmetries with 3 cycles can be represented by $12n^3$. The 3 rotational symmetries with 4 cycles can be represented by $3n^4$ and the 8 rotational symmetries with 2 cycles can be represented by $8n^2$. The identity symmetry is represented by n^6 since there are six faces on a cube. We have to add all of these and divided by the number of symmetries to get the number of distinct cubes with n colors and only applying rotational symmetries. We get the following formula.

$$\frac{n^6 + 3n^4 + 12n^3 + 8n^2}{24}$$

A cube does not only have faces. We can continue looking at a cube to find the formula for distinct cube when coloring the vertices and the edges.

There are a total of 8 vertices for a cube. We already know that there are 9 rotations that have axes through the center of opposite faces, 6 that have axes through the center of opposite edges and 8 that have axes through opposite vertices. The first 9 are comprised of 90, 180, and 270 degree rotations. The 90 degree and 270 degree rotations have 2 cycles and the 180 degree rotations have 4 cycles. There are three of each totaling 9 and will look like the following.

(1234)(5678) (13)(24)(57)(68) (1432)(5876)

The 6 rotations about the axes through opposite edges will be similar to the following.

(14)(28)(35)(67)

The 8 rotation the come from rotation about the axes through opposite vertices will be similar to the following.

(1)(7)(245)(386) (1)(7)(254)(368)

We see that from the above rotations, 4 of them have 4 cycles and 2 of them have 2 cycles. Since we are looking at cube rotations we know that there will be 4 more of the 2 cycle rotations giving us a total of 6. There will be 13 more of the 4 cycle rotations giving us a total of 17. Thus, we can get the following formula.

$$\frac{n^8 + 17n^4 + 6n^2}{24}$$

We can also look at the number of edges and where the rotations take the edges. We know that there are 12 edges on a cube. We can do a similar process to the one above to find our formula. The rotations that have axes through the faces are comprised of 90, 180, and 270 degree rotations. The 90 degree and 270 degree rotations have 3 cycles and the 180 degree rotations have 6 cycles. They will look like the following.

$(1234)(5678)(9, 10, 11, 12)$ $(13)(24)(57)(68)(9, 11)(10, 12)$ $(1432)(5876)(9, 12, 11, 10)$

The rotations that have axes through the edges look like the following rotation.

$(1)(7)(29)(35)(4, 10)(6, 12)(8, 11)$

The rotations that have axes through the vertices look like the following rotations.

$(149)(2, 12, 5)(38, 10)(6, 11, 7)$ $(194)(25, 12)(3, 10, 8)(67, 11)$

There are 2 rotations that have 3 cycles, 2 rotations that have 4 cycles, one rotation that has 6 cycles and one rotation that has 7 cycles. When we take into account the rotations that are not shown we end up with 6 rotations that have 3 cycles, 8 rotations that have 4 cycles, 3 rotations that has 6 cycles and 6 rotations that has 7 cycles. When we include the identity we get the following formula.

$$\frac{n^{12} + 6n^7 + 3n^6 + 8n^4 + 6n^3}{24}$$

.

Our goal is to take this method and use it for 4 dimensional shapes. To get more practice with this method we looked at the other four platonic solid. The other platonic solids are a tetrahedron, an octahedron, a dodecahedron and an icosahedron.

The tetrahedron has 12 rotational symmetries and 4 faces. They are as follows.

ρ_0: (1) (2) (3) (4)

ρ_1: (1) (234) ρ_2: (1) (243)

ρ_3: (2) (134) ρ_4: (2) (143)

ρ_5: (3) (124) ρ_6: (3) (142)

ρ_7: (4) (123) ρ_8: (4) (132)

ρ_9: (14) (23) ρ_{10}: (12) (34) ρ_{11}: (13) (24)

Every rotation has 2 cycles except the identity. Thus, for n colors the order the colored tetrahedrons that are the same when these rotations are applied to them is $11n^2$. We can add this to the order of the colored tetrahedrons when the identity is applied to them (n^4) to get $n^4 + 11n^2$. We divide this by the number of symmetries (12) and get the number of distinct tetrahedrons with n colors up to rotational symmetries. This number can be represented by the following formula.

$$\frac{n^4 + 11n^2}{12}$$

We also found how these rotations would affect our vertices and edges. There are four vertices and each one of them corresponds to one of the four faces. Since the vertices are always across from the corresponding angles the rotations affect them the same way as the faces. This means that the formula for the number of distinct tetrahedrons when coloring the vertices is

$$\frac{n^4 + 11n^2}{12}$$

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The edges of a tetrahedron are different. There are six edges on a tetrahedron. When we look at the rotational symmetries of a tetrahedron we see that there are four that rotate 120 degrees, 4 that rotate 240 degrees and three the rotate 180 degrees (not including the identity). When we look at how these rotations affect the edges we get that all the rotations that are 120 degrees or 240 degrees have 2 cycles and the three that are 180 degrees have 4 cycles. This gives us a total of 8 rotations with 2 cycles and three rotations with 4 cycles. When we include the identity we can get the following formula.

$$\frac{n^6 + 3n^4 + 8n^2}{12}$$

Next, we will look at the octahedron. The octahedron has the same amount of rotational symmetries as the cube (24) and has 8 faces. The rotational symmetries are the following.

ρ_0: (1) (2) (3) (4) (5) (6) (7) (8)

ρ_1: (1234) (5678) ρ_2: (13) (24) (57) (68) ρ_3: (1432) (5876)

ρ_4: (2673) (1584) ρ_5: (27) (63) (18) (54) ρ_6: (2376) (1485)

ρ_7: (1265) (4378) ρ_8: (16) (25) (47) (38) ρ_9: (1562) (4873)

ρ_{10}: (17) (26) (35) (48) ρ_{11}: (15) (28) (37) (46)

ρ_{12}: (28) (65) (71) (34) ρ_{13}: (21) (64) (78) (45)

ρ_{14}: (46) (32) (71) (85) ρ_{15}: (41) (35) (76) (82)

ρ_{16}: (1) (7) (453) (628) ρ_{17}: (1) (7) (435) (682)

ρ_{18}: (5) (3) (186) (247) ρ_{19}: (5) (3) (168) (274)

ρ_{20}: (8) (2) (163) (457) ρ_{21}: (8) (2) (136) (475)

ρ_{22}: (4) (6) (138) (527) ρ_{23}: (4) (6) (183) (572)

The formula for n colors for the octahedron is

$$\frac{n^8 + 17n^4 + 6n^2}{24}$$

We noticed that symmetries with similar axes had the same number of cycles. For example the tetrahedron has four axes of symmetry that go from the center of one face to the opposite vertex. All of the rotational symmetries around these axes have the same number of cycles. This is true for all similar axes for platonic solids. Knowing this we do not need to find all of the rotational symmetries of the dodecahedron and the icosahedron. We only need to find the symmetries for one of every type of rotational axis.

The dodecahedron has three types of rotational symmetry axis. Six that go from the center of a face to the opposite face, fifteen that go from the center of an edge to the opposite edge and 10 that go from a vertex to the vertex that is opposite. There are 12 faces on a dodecahedron and 60 rotational symmetries. We will first look at an axis that goes from the center of a face to the center of the opposite face. There are four rotational symmetries around an axis like this. The following is one of those groups of four.

ρ_1: (1) (7) (23456) $(89, 10, 11, 12)$ ρ_2: (1) (7) (24635) $(8, 10, 12, 9, 11)$

ρ_3: (1) (7) (25364) $(8, 11, 9, 12, 10)$ ρ_4: (1) (7) (26543) $(8, 12, 11, 10, 9)$

Since these 4 symmetries all have 4 cycles each one can be represented as n^4 where n is the number of colors the dodecahedron is being painted with. There are five other axes similar to this one so all 24 of the symmetries can be represented by $24n^4$. Next we will look at an axis that goes from the center of an edge to the center of the opposite edge.

ρ_5: (12) (36) $(4, 12)$ (58) $(7, 10)$ $(9, 11)$

There are 6 cycles in this symmetry and 14 other symmetries that are similar. The order of the dodecahedrons that stay the same when ρ_5 is applied is n^6 for n colors. Since there are 14 other symmetries like this one the order of the dodecahedrons that stay the same for all 15 is $15n^6$. Next, we will look at the axis of symmetry that goes from vertex to opposite vertex.

ρ_6: (126) $(3, 12, 5)$ $(48, 11)$ $(7, 10, 9)$ ρ_7: (162) $(35, 12)$ $(4, 11, 8)$ $(79, 10)$

These two symmetries have 4 cycles and there are 9 other axis of symmetry. The other 9 have 2 symmetries and all of those symmetries have 4 cycles. The order of the dodecahedrons that stay the same when these symmetries are applied is $20n^4$ where n is the number of colors. We can add these orders to the order of the dodecahedrons that stay the same when the identity is applied (n^{12}) and then divide the sum by the number of rotational symmetries. We get the following equation for the number of distinct dodecahedrons with n colors up to rotational symmetries.

$$\frac{n^{12} + 15n^6 + 44n^4}{60}$$

The icosahedron is dual to the dodecahedron. When shapes are dual, the vertices of one of the shapes correspond to the faces of the other shape. The edges connecting the vertices of one of the shapes correspond the the edges connecting the faces of the other shape. These two correspondences give the shapes the same symmetry groups, because the correspondences have to hold always. This means that the dodecahedron also has 60 rotational symmetries. The icosahedron has 20 faces. We will look at the axes of symmetry that go from a vertex to the opposite vertex. One of the axes has the following rotations.

ρ_1: $(189, 10, 2)$ $(357, 18, 20)$ $(46, 17, 19, 12)$ $(11, 13, 14, 15, 16)$ ρ_2: $(1928, 10)$ $(37, 20, 5, 18)$ $(4, 17, 12, 6, 19)$ $(11, 14, 16, 13, 15)$

ρ_3: $(1, 10, 829)$ $(3, 18, 5, 20, 7)$ $(4, 19, 6, 12, 17)$ $(11, 15, 13, 16, 14)$ ρ_4: $(12, 10, 98)$ $(3, 20, 18, 75)$ $(4, 12, 19, 17, 6)$ $(11, 16, 15, 14, 13)$

There are 5 more axes with similar rotations to make a total of 24 rotational about axes that go from vertex to opposite vertex. The order of icosahedrons that stay the same when these rotations are applied can be represented by $24n^4$. The next group of axes are the ones that go from the center of one edge to the center of the opposite edge. There are 15 of them. All of their permutation notations are similar to the following one.

ρ_5: (12) (38) (49) $(5, 10)$ $(6, 20)$ $(7, 13)$ $(12, 17)$ $(14, 18)$ $(15, 19)$ $(16, 11)$

The order of the icosahedrons that stay the same when these rotations are applied can be represented by $15n^{10}$. The last type of axes for the icosahedron are the ones that go from the center of one face to the center of the opposite face. The rotations are similar to the following rotations.

ρ_6: (1) (11) (285) (396) $(4, 10, 7)$ $(12, 18, 15)$ $(13, 19, 16)$ $(14, 20, 17)$ ρ_7: (1) (11) (258) (369) $(47, 10)$ $(12, 15, 18)$ $(13, 16, 19)$ $(14, 17, 20)$

There are 9 other axes like this one. Thus, the order of the icosahedrons that stay the same when these rotations are applied can be represented by $20n^8$. The order of the icosahedrons that stay the same when the identity is applied can be represented by n^{20}. We can take the sum of these orders and divide by the number of rotations. We get the following formula for the number of distinct icosahedrons with n colors up to rotational symmetries.

$$\frac{n^{20} + 20n^8 + 15n^{10} + 24n^4}{60}$$

4 Exploring the Fourth Dimension

Now that we have a good understanding of how to use Burnside's formula we will try to understand how colorings work in the fourth dimension. There are many ways of trying to think about the fourth dimension. A popular one is dimensional analogy. This is where we try to imagine how a 2-dimensional being would see a 3-dimensional being and then apply that to how we would see a 4-dimensional being. A 3-dimensional being moving through a 2-dimension world would, to a 2-dimensional being, look like a constantly changing 2-dimensional shape if it was not a uniform shape. An example of this would be a sphere moving through a plane. When the sphere first touches the plane there is only a point on the plane. As the sphere moves through the plane, the cross section is a circle that gets bigger and then smaller until it is a point again. A 4-dimensional being moving through our 4-dimensional world would look like a 3-dimensional being that is oddly changing size (Rucker 1976). Another way is with perspective projection. This is how our eyes view the world. At any given time we see 2 dimensional shapes but because we can move around and because of depth and lighting we perceive them as 3-dimensional shapes. When we draw 4-dimensional shapes on paper we can use different shading to help us understand what part of the shape is further away in the fourth dimension (O'Conner 2003).

We have spent a lot of time trying to understand fourth dimensional symmetries. We will look at a 5-cell. This is the fourth dimensional tetrahedron. It is made of 5 tetrahedrons which are its cells. A tetrahedron has 8 rotational symmetries where one side stays the same. This triangular side is rotating how a triangle would rotate but it does not go to another face. Using the idea of dimensional analogy we can find some rotational symmetries of the 5-cell. One of the cells will not go to any of the others but can still perform all of its rotational symmetries. A tetrahedron has 11 rotational symmetries that are not the identity. Since there are 5 different cells (tetrahedrons) and 11 rotational symmetries not including the identity of a cell, a 5-cell has 55 rotational symmetries of a certain type not including the identity.

We put this hypothesis to the test and found that we do not get 55 different rotational symmetries. Some of the symmetries repeat themselves so we get 36 rotational symmetries including the identity. They are as follows.

(1)(2)(3)(4)(5) (1)(4)(523) (1)(4)(532) (1)(5)(342)

(1)(5)(324) (1)(3)(542) (1)(3)(524) (1)(2)(543) (1)(2)(534)

(1)(34)(52) (1)(35)(42) (1)(45)(23) (2)(5)(314) (2)(5)(341)

(2)(4)(531) (2)(4)(513) (2)(3)(451) (2)(3)(415) (2)(34)(51)

(2)(31)(45) (2)(35)(41) (3)(4)(125) (3)(4)(152) (3)(5)(124)

(3)(5)(142) (3)(54)(21) (3)(42)(15) (3)(41)(52) (4)(5)(123)

(4)(5)(132) (4)(51)(23) (4)(53)(12) (4)(52)(13) (5)(12)(34)

(5)(14)(23) (5)(13)(24)

We know that the order of the group of rotational symmetries of a 5-cell with be half the order of the group of total symmetries of a 5-cell. Since there are 120 total symmetries of a 5-cell there are 60 rotational symmetries of a 5-cell. This means we need to find the other 24 symmetries. We can do this by composing two rotational symmetries that we already have together. Since the symmetries we already have are even the composition of those symmetries are even. When we compose the 36 symmetries that we already have we can get the following rotational symmetries.

(14352) (14523) (12354) (15342) (14235) (12345) (15423) (14253)

(12543) (12435) (14325) (15324) (13524) (15432) (13254) (13425)

(12534) (14532) (15243) (13245) (15234) (13542) (12453) (13452)

This list of 24 rotational symmetries combined with the list of 36 rotational symmetries gives us our 60 rotational symmetries of a 5-cell. These 60 are the group of all even permutations. Now that we know the rotational symmetries we can use burnside's formula to find the number of distinct 5-cells when coloring the cells n colors.

The number of 5-cells that stay the same when the identity is applied with n colors is n^5. We have 35 symmetries that have 3 cycles. The number of 5-cells that stay the same when those 35 symmetries are applied with n colors is $35n^3$. Then we have 24 symmetries that only have one cycle. The number of 5-cells that stay the same when those 24 symmetries are applied with n colors is $24n$. When we add these together and divide by the number of rotational symmetries we get the following equation for the number of distinct 5-cells with n colors.

$$\frac{n^5 + 35n^3 + 24n}{60}$$

.

We also decided to see where each rotation would take the vertices, edges and faces of the 5-cell. The vertices have the exact same formula as the cells. This is because each cell has one corresponding vertex that is opposite it. The edges and faces are different.

Not including the identity there are 3 different types of rotation the accompany the 5-cell. These rotations are the ones where two cells stay the same, one cell stays the same, and every cell goes to another. There are 20 where two cells stay the same, 15 where one cell stays the same and 24 where all cells go to other cells. If we find where one of each of these rotations take the edges and the faces we can create formulas for coloring the edges and faces.

There are 10 edges on a 5-cell. The rotations previously mentioned give us cycles like the following

$(123)(456)(789)(10)$ $(1)(6)(34)(25)(78)(9, 10)$ $(17248)(39, 10, 56)$

Knowing what each cycle looks like and knowing how many of each different cycle there is we can come up with the following formula.

$$\frac{n^{10} + 15n^6 + 20n^4 + 24n^2}{60}$$

The formula for the faces is exactly the same. This may be attributed to the fact that every face of the 5-cell has a corresponding edge. When the 5-cell rotates the faces and edges rotate in a similar fashion because of the correspondence between them. The following three cycles are the three types of rotations of a 5-cell and where they take each face.

$(1)(234)(5, 8, 10)(697)$ $(2)(9)(15)(46)(37)(8, 10)$ $(4, 2, 6, 8, 10)(17359)$

Knowing what each cycle looks like and knowing how many of each different cycle there is we can come up with a formula. The formula is the same as the previous one.

5 Conclusion

Our goal was to find a connection between similar shapes of different dimensions using Burnside's formula. We were able to come up with a variety of different formulas for the number of distinct shapes when colored with n colors. In order to do this we had to find all of the rotational symmetries for these different shapes and write them in cycle notation. For many of the shapes that we looked at we noticed that the rotational symmetries were the even permutations. This made finding all of the rotational symmetries of those shapes easier. For shapes where the rotational symmetries were not all even, like the 4-cube, the problem becomes more difficult. The number of rotational symmetries of a 4-cube is easy to find but there are not listings of them in cycle notation. This means we have to find them ourselves. When we find an odd permutation we do not immediately know if it is a rotational symmetry or not. For shapes that only have even permutations for rotational symmetries, we know if a permutation is a rotation or not. After finding all of our generalized formulas we compared the formulas. When we compared the formulas of similar shapes in different dimensions, such as the tetrahedron and the 5-cell, we could not find any similarity between their formulas.

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