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Senior Capstone project

Applications of Regular Markov Chains

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Acknowledgments

I would like to give thanks to my mentor Dr. Jebessa Mijena who gave me the invaluable opportunity to research the topic of Markov chains, which led me to give special interest to the applications of regular Markov chains. Secondly, i would also like to thank my family and friends who helped me to stay focused on my research. Finally, I would like to give thanks to all the faculty at Georgia College and State University for being there for me in the good and bad times.

Contents

Abstract

Andrey Markov developed the idea of a Markov chain to model a stochastic process that has the memorylessness property, i.e., the probability of each future event depends only on the previous event. My research focuses on the applications of a particular type of a Markov chain called a regular Markov chain. Throughout this discussion I will briefly discuss the stochastic process and how a Markov chain could be used to model it, time homogeneity, n-step transition probabilities, stationary and limiting distribution, and finally how I use this to analyze 21 seasons of the soccer team Real Madrid and all of this last year weather in Milledgeville to get future projections of both. In particular, I had to derive a state space for both Real Madrid and the weather of Milledgeville that captured all possible states that they could be in. The state space for Real Madrid I described them as win, draw, or a loss, and as for the weather in Milledgeville I described the states as sunny, cloudy, or rainy. From this point I analyzed the transitions between each state to derive a transition probability matrix for both Real Madrid and the weather of Milledgeville, so that I could the n-step transition probability theorem to give future projections of Real Madrid and the weather of Milledgeville.

1 Introduction

Through my exploration of data analysis using various methods like parameter estimation, regression, and the Markov process; using the Markov process takes the cake. The following will guide you through the ideas that I, and Dr. Mijena went through to analyze the soccer team Real Madrid and the weather of Milledgeville. I certainly hope that you enjoy this as much as me and Dr. Mijena did; it was a truly eye opening adventure for the both of us.

2 Theory of Regular Markov Chains

2.1 Conditions of a Regular Markov Chain

Before we get started, I want to let you know that me and Dr. Jebessa Mijena researched more than just regular Markov chains in discrete time, we also researched other types of Markov chains, but all of which have the following in common except the Markov chains that have memory, but nonetheless you use the same conditions but with a smarter way of looking at the state space. At any rate, the two main conditions we need for a regular Markov chain goes as follows:

- A discrete-time stochastic process is a Markov chain with a finite state space if for $n = 0, 1, 2, ...$

$$P[X_n = i_n | X_{n-1} = i_{n-1}, ..., X_0 = i_0] = P[X_n = i_n | X_{n-1} = i_{n-1}]$$

  this is better known as the memorylessness property, since all that matters is current state. This can be checked using the chi square test for independence, since we would need to show that the future is independent of the past given the present. Another extremely important idea is

- A discrete-time Markov chain is a time homogeneous Markov Chain if

$$P[X_n = j | X_{n-1} = i] = P[X_{n-1} = j | X_{n-2} = i] = ... = p_{ij},$$

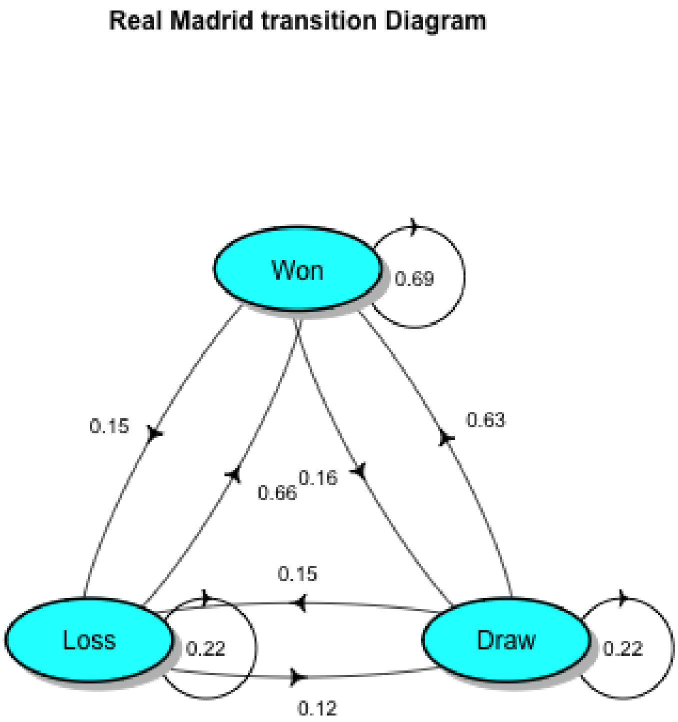
which are the transition probabilities, and also

$$\sum_{j=1}^{N} p_{ij} = 1, \text{ where } N \text{ is the number of states.}$$

Furthermore, if all the transition probabilities are positive, then the Markov chain is considered to be regular. In order to verify time homogeneity we can use a chi square goodness of fit test, so that we can show that the transition probabilities do not change with respect to time.

2.2 Brief Explanation of Transition Diagrams

Another way of interpreting transition probabilities is through a graph of our 1-step transition probabilities, which we call a transition probability diagram. The states are indicated with some kind of shape, e.g., circles and the numbers with the arrows indicate the probability of going from one state to another, or stay in the current state. For example, the probability of going from a win to a draw is 0.16, going from a loss to a win is 0.66, going from draw to a loss is 0.15



2.3 n-Step Transition Probability Theorem

The two conditions for a regular Markov chain gives way for a powerful tool that we can use to make projections for regular Markov chains, which is called the n-Step Transition Probability theorem:

$$\begin{bmatrix} P[X_n = 1] & . & . & . & P[X_n = N] \end{bmatrix} = \bar{\phi} P^n,$$

where $\bar{\phi} = \begin{bmatrix} P[X_0 = 1] & P[X_0 = 2] & . & . & . & P[X_0 = N] \end{bmatrix}$, and

$$P = \begin{bmatrix} p_1 & p_2 & . & . & . & p_N \\ p_1 & p_2 & . & . & . & p_N \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ p_1 & p_2 & . & . & . & p_N \end{bmatrix}$$

where P_{ij} is the one step transition probability going from state i to state j.

Now lets consider the proof that goes behind this seemingly simple n-step transition probability theorem. Before we do so, I want to give you the design of the proof, which is to prove the theorem by induction with the help of a lemma describing the individual probabilities of going from state to state. Let's get this party started! So, from this point and on, let $\phi(i) = P[X_0 = i]$ and the stochastic row vector

$$\bar{\phi} = \begin{bmatrix} P[X_0 = 1] & P[X_0 = 2] & . & . & . & P[X_0 = N] \end{bmatrix},$$

so $\phi(i)$ is the i-th entry of $\bar{\phi}$. Now let us prove the following lemma for discrete-time homogeneous Markov chains,

$$P[X_n = j] = \sum_{i=1}^{N} P[X_{n-1} = i] \cdot P_{ij}$$

Let

$$S = \bigcup_{i=1}^{N} (X_{n-1} = i).$$

(In layman's terms S is the collection of random variables in the present state). Now consider $(X_n = j) \cap S$. Note that if $A \subseteq S$, then $A = A \cap S$, so since $\{X_n = j\} \subset S$ (this is only possible if the state space captures all of the possible events.),

$$\begin{aligned}(X_n = j) &= (X_n = j) \cap S \\ &= (X_n = j) \cap \bigcup_{i=1}^{N} (X_{n-1} = i) \\ &= \bigcup_{i=1}^{N} [(X_n = j) \cap (X_{n-1} = i)]\end{aligned}$$

Since $(X_n = j) \cap (X_{n-1} = i)$ where $i = 1, 2, \cdots$, are mutually exclusive events (must go to another state, or stay where it is at), we know

$$\begin{aligned}P[X_n = j] &= P\left[\bigcup_{i=1}^{N} (X_n = j) \cap (X_{n-1} = i)\right] \\ &= \sum_{i=1}^{N} P[(X_n = j) \cap (X_{n-1} = i)] \\ P[X_n = j] &= \sum_{i=1}^{N} P[X_{n-1} = i] \cdot P[X_n = j | X_{n-1} = i]\end{aligned}$$

So, for a time homogeneous Markov Chain in discrete time we can further simplify $P[X_n = j]$,

$$P[X_n = j] = \sum_{i=1}^{N} P[X_{n-1} = i] \cdot p_{ij}$$

Now lets prove $P[X_n = j] = \bar{\phi} P^n$ by using mathematical induction. For $n = 1$,

$$P[X_1 = j] = \sum_{i=1}^{N} P[X_0 = i] \cdot p_{ij} = \sum_{i=1}^{N} \phi(i) \cdot p_{ij}$$

Since $\phi(i)$ is the i-th entry of the row vector

$$\bar{\phi} = \begin{bmatrix} P[X_0 = 1] & P[X_0 = 2] & . & . & . & P[X_0 = N] \end{bmatrix}$$

and p_{ij} is the i-th entry of the j-th column of

$$P = \begin{bmatrix} p_1 & p_2 & . & . & . & p_N \\ p_1 & p_2 & . & . & . & p_N \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ p_1 & p_2 & . & . & . & p_N \end{bmatrix}$$

then by the definition of matrix multiplication, $P[X_1 = j]$ is the j-th entry of $\bar{\phi}P$. Now let's consider $P[X_{n+1} = j]$. Thus,

$$P[X_{n+1} = j] = \sum_{i=1}^{N} P[X_n = i] \cdot p_{ij}$$

Assuming the inductive hypothesis that $P[X_n = i]$ is the i-th entry of the row vector $\bar{\phi}P^n$ and p_{ij} is the i-th entry of the j-th column of

$$P = \begin{bmatrix} p_1 & p_2 & . & . & . & p_N \\ p_1 & p_2 & . & . & . & p_N \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ p_1 & p_2 & . & . & . & p_N \end{bmatrix}$$

then by the definition of matrix multiplication, $P[X_{n+1} = j]$ is the j-th entry of $(\bar{\phi}P^n)P = \bar{\phi}P^{n+1}$. Therefore, by mathematical induction, the probability distribution of X_{n+1} is given by $\bar{\phi}P^{n+1}$. ■

2.4 Stationarity

Now that we've proven the n-Step Transition Probability theorem I would like to show you another way of writing $\begin{bmatrix} P[X_n = 1] & . & . & . & P[X_n = N] \end{bmatrix} = \bar{\phi}P^n$ which is $S_0 \cdot P^n = S_n$ where S_0 is $\bar{\phi}$ and S_n is the n-step distribution. Furthermore, we can write S_n in terms of the previous step. Since $S_{n-1} = S_0 \cdot P^{n-1}$, then

$$\begin{aligned} S_n &= (S_0 \cdot P^{n-1})P \\ S_n &= S_{n-1} \cdot P \end{aligned}$$

This gives way for another extremely useful idea about stationarity (a way of describing how many of what in the long run), which is if a time homogeneous Markov chain has a **stationary distribution** S, then

$$S \cdot P = S.$$

If a time homogeneous Markov chain has a stationary distribution S, then S is unique. The proof goes by contradiction. So, let's assume there exist S and S' that are both stationary distributions of a given time homogeneous Markov chain. Then S and S' must occur simultaneously, since if they didn't, then one of S and S' wouldn't be stationary. But since $S_n = S_{n-1} \cdot P$ yields a unique set of probabilities, then S and S' can't occur simultaneously unless $S = S'$. Therefore the stationary distribution of a time homogeneous Markov chain must be unique. ■

If a time homogeneous Markov chain has a stationary distribution S, then

$$S \cdot P^n = S.$$

The proof goes by using mathematical induction. For $n = 1$, $S \cdot P^n = S$ gives $S \cdot P = S$, which is true by the definition of the stationary distribution of a time homogeneous Markov chain. Now let's consider $S \cdot P^{n+1}$. Thus,

$$S \cdot P^{n+1} = S \cdot P^n \cdot P.$$

Assuming the inductive hypothesis that $S \cdot P^n = S$, we have

$$\begin{aligned} S \cdot P^{n+1} &= S \cdot P^n \cdot P \\ &= S \cdot P \\ S \cdot P^{n+1} &= S, \text{ since } S = S \cdot P \end{aligned}$$

Therefore, by mathematical induction, if a time homogeneous Markov chain has a stationary distribution S, then $S \cdot P^n = S$. ■

2.5 Limiting Matrix

Another extremely important idea is if a discrete time Markov chain has **limiting matrix** $\bar{P}$, then

$$\bar{P} = \lim_{n\to\infty} P^n$$

Furthermore, we can generalize the limiting form of a regular Markov chain, All regular Markov chains, i.e., all the transition probabilities are positive, have a limiting matrix of the form

$$\bar{P} = \lim_{n\to\infty} P^n = \begin{bmatrix} A_{11} & A_{12} & . & . & . & A_{1N} \\ A_{21} & A_{22} & . & . & . & A_{2N} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ A_{N1} & A_{N2} & . & . & . & A_{NN} \end{bmatrix},$$

where N is the number states and $\begin{bmatrix} A_{11} & A_{12} & . & . & . & A_{1N} \end{bmatrix}$ is the stationary distribution of a regular Markov chain. Before we can prove this we must show that if a time homogeneous Markov chain has a stationary distribution S and a limiting matrix $\bar{P}$, then $S \cdot \bar{P} = S$.

Since we're considering a time homogeneous Markov chain, then $S \cdot \bar{P} = S$ is a corollary of the theorem $S \cdot P^n = S$, since n can be arbitrarily large in the expression of $S \cdot P^n = S$. Thus, $S \cdot \bar{P} = S$ for a time homogeneous Markov chain ■

The proof of the limiting form theorem goes by letting the ij-th entry of $\bar{P}$ be A_{ij}

and supposing that the stationary distribution S and $\bar{P}$ exists, and using the fact that $\bar{P}^2 = \bar{P}$. Thus,

$$\begin{bmatrix} A_{11} & A_{12} & . & . & . & A_{1N} \\ A_{21} & A_{22} & . & . & . & A_{2N} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ A_{N1} & A_{N2} & . & . & . & A_{NN} \end{bmatrix}^2 = \begin{bmatrix} A_{11} & A_{12} & . & . & . & A_{1N} \\ A_{21} & A_{22} & . & . & . & A_{2N} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ A_{N1} & A_{N2} & . & . & . & A_{NN} \end{bmatrix}$$

From this point let's consider the i-th row entries of $\bar{P}$ using equation $\bar{P}^2 = \bar{P}$. Thus,

$$\begin{aligned} A_{i1}A_{11} + A_{i2}A_{21} + \ldots + A_{iN}A_{N1} &= A_{i1} \\ A_{i1}A_{12} + A_{i2}A_{22} + \ldots + A_{iN}A_{N2} &= A_{i2} \\ &. \\ &. \\ &. \\ A_{i1}A_{1N} + A_{i2}A_{2N} + \ldots + A_{iN}A_{NN} &= A_{iN} \end{aligned}$$

$$\begin{bmatrix} A_{i1} & . & . & . & A_{iN} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & . & . & . & A_{1N} \\ A_{21} & A_{22} & . & . & . & A_{2N} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ A_{N1} & A_{N2} & . & . & . & A_{NN} \end{bmatrix} = \begin{bmatrix} A_{i1} & . & . & . & A_{iN} \end{bmatrix}$$

Since

$$\bar{P} = \begin{bmatrix} A_{11} & A_{12} & . & . & . & A_{1N} \\ A_{21} & A_{22} & . & . & . & A_{2N} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ A_{N1} & A_{N2} & . & . & . & A_{NN} \end{bmatrix}$$

then $\begin{bmatrix} A_{i1} & . & . & . & A_{iN} \end{bmatrix} \cdot \bar{P} = \begin{bmatrix} A_{i1} & . & . & . & A_{iN} \end{bmatrix}$.

So, since we're considering a time homogeneous Markov chain that has a stationary distribution S and a limiting matrix $\bar{P}$, then it must follow by $S \cdot \bar{P} = S$ being unique and

$$\begin{bmatrix} A_{i1} & . & . & . & A_{iN} \end{bmatrix} \cdot \bar{P} = \begin{bmatrix} A_{i1} & . & . & . & A_{iN} \end{bmatrix},$$

$S = \begin{bmatrix} A_{i1} & . & . & . & A_{iN} \end{bmatrix}$. In particular, each row of $\bar{P}$ must be equal to S, since S is unique. Thus, the limiting form of a regular Markov chain has been proven. ■

3 Data Analysis

Now that we have a firm understanding of a regular Markov chains, let's dive into the data analysis that me and Dr. Jebessa Mijena worked so hard on. First we analyzed the soccer team Real Madrid which currently holds the record on 13 European Cup titles, and 33 La Liga titles, 25 international titles.

3.1 Estimating the Transition Probability Matrix for Real Madrid

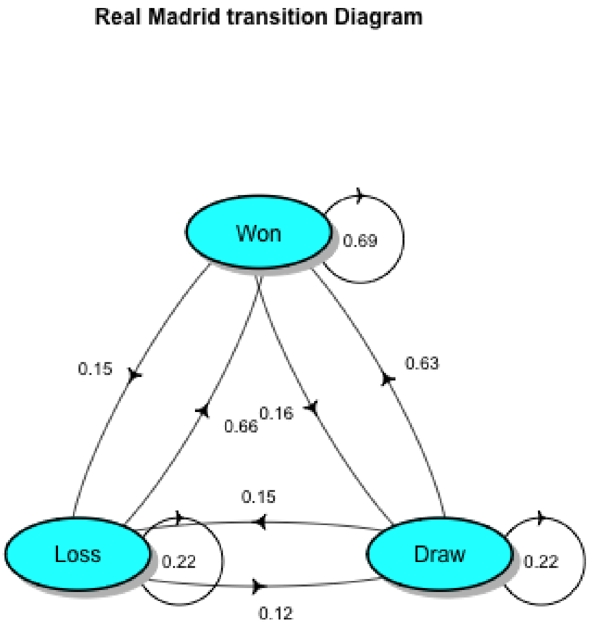
Using **R** we were able estimate the transition probabilities using bootstrapping with $nboot = 20000$. From this point, we showed that the soccer team Real Madrid has the conditions for a regular Markov chain for 2000-2018 seasons. Initially, we analyzed 21 seasons, the Markov property was good for all the seasons, since the chi square statistic yielded a p-value of 0.99, but since 1996-2000 seasons didn't have the time homogeneous property, so we had to only consider 2000-2018 seasons.

Seasons	P-Value	Seasons	P-Value
1997-2018	0.00	2007-2018	0.40
1998-2018	0.01	2008-2018	0.32
1999-2018	0.03	2009-2018	0.74
2000-2018	0.12	2010-2018	0.72
2001-2018	0.07	2011-2018	0.65
2002-2018	0.11	2012-2018	0.85
2003-2018	0.31	2013-2018	0.66
2004-2018	0.32	2014-2018	0.28
2005-2018	0.20	2015-2018	0.41
2006-2018	0.38	2016-2018	0.21

Table 1: Time Homogeneity

The p-values less than 0.05 means not time homogeneous.

Then we were able to compile this beautiful transition probability diagram using the data from 2000-2018 seasons



From the transition probability diagram we can easily derive the transition probability matrix by examining the arrows between the states, since the arrows indicate the probability of going from one state to another, or stay in the current state. From that point we label a 3×3 matrix with letters, or words, to indicate the states. For example, the probability of going from W to D is 0.16, going from L to W is 0.66, going from D to L is 0.15, etc. Please note that it doesn't matter how you label your transition probability matrix as long you label the left, or right, side the same as label the top side, i.e., top to bottom and left to right, or vice versa. In the following transition probability matrix I labeled it so that the letters on the left side of the matrix indicates the present state and the letters on the top side indicate the future state

$$\begin{array}{c} \\ W \\ L \\ D \end{array} \begin{array}{c} \begin{array}{ccc} W & L & D \end{array} \\ \begin{bmatrix} 0.69 & 0.15 & 0.16 \\ 0.66 & 0.22 & 0.12 \\ 0.63 & 0.15 & 0.22 \end{bmatrix} \end{array}$$

3.2 Standard Error of the Bootstrap Estimate of the Transition Probability Matrix for Real Madrid

We used **R** to determine the standard error for each of the transition probabilities in our bootstrap estimate of the transition probability matrix for Real Madrid,

$$\begin{array}{c} \\ W \\ L \\ D \end{array} \begin{array}{c} \begin{array}{ccc} W & L & D \end{array} \\ \begin{bmatrix} 0.0002 & 0.0001 & 0.0001 \\ 0.0003 & 0.0003 & 0.0002 \\ 0.0003 & 0.0002 & 0.0003 \end{bmatrix} \end{array}$$

which is really nice, since we know that it is very unlikely that the transition probabilities will exceed three standard deviations beyond the mean. Furthermore, we can use the SE (standard error matrix) to calculate confidence intervals, e.g., the 95% confidence interval for the true mean of all the transition probabilities for Real Madrid is determined by $\hat{P} \pm 1.96 \cdot \mathrm{SE}$, where $\hat{P}$ is our bootstrap estimate for the transition probability matrix for Real Madrid. Since all the elements of SE is considerably small, i.e., each element is less than or equal to 0.0003, where $\hat{P}$ is our bootstrap estimate for the transition probability matrix for Real Madrid and SE is the standard error matrix for our bootstrap estimate of our transition probability matrix for Real Madrid, then $\hat{P} \pm 1.96 \cdot \mathrm{SE} \approx \hat{P}$, i.e.,

$$\begin{bmatrix} \mu_{p_{WW}} & \mu_{p_{WL}} & \mu_{p_{WD}} \\ \mu_{p_{LW}} & \mu_{p_{LL}} & \mu_{p_{LD}} \\ \mu_{p_{DW}} & \mu_{p_{DL}} & \mu_{p_{DD}} \end{bmatrix} \approx \begin{bmatrix} 0.6931 & 0.1489 & 0.1580 \\ 0.6650 & 0.2192 & 0.1158 \\ 0.6258 & 0.1546 & 0.2196 \end{bmatrix} \approx \hat{P},$$

In particular, since the standard error for a win to a win is 0.0003, i.e., then we're 95% confident that the true mean of the transition probability for a win to a win is approximately 0.69.

3.3 Projections for Real Madrid for 2018-2019 Season

Now that we've established a good estimate of the transition probability matrix for the weather of Milledgeville and it has the properties of a regular Markov chain, we can use the n-transition probability theorem to predict what Real Madrid will do in the future.

The last game of the 17/18 season R. Madrid had a draw, so let's consider $P[X_{49} = L|X_0 = D]$. Using the n-step transition probability theorem,

$$\begin{aligned} S_{49} &= S_0 \cdot P^{49} \\ & \quad\;\; \begin{matrix} W & \;\; L & \;\; D \end{matrix} \\ &\approx \begin{bmatrix} 0.68 & 0.16 & 0.16 \end{bmatrix} \\ P[X_{49} = L|X_0 = D] &\approx 0.16 \end{aligned}$$

Furthermore, we notice $S_{50} = \begin{bmatrix} 0.68 & 0.16 & 0.16 \end{bmatrix}$ which is S_{49}, so $\begin{bmatrix} 0.68 & 0.16 & 0.16 \end{bmatrix}$ is the stationary distribution, so by multiplying the stationary distribution by the total number of games in a season, which is 38 games, we get that Real Madrid is projected to win 26, lose 6, and draw 6 in 18/19 season.

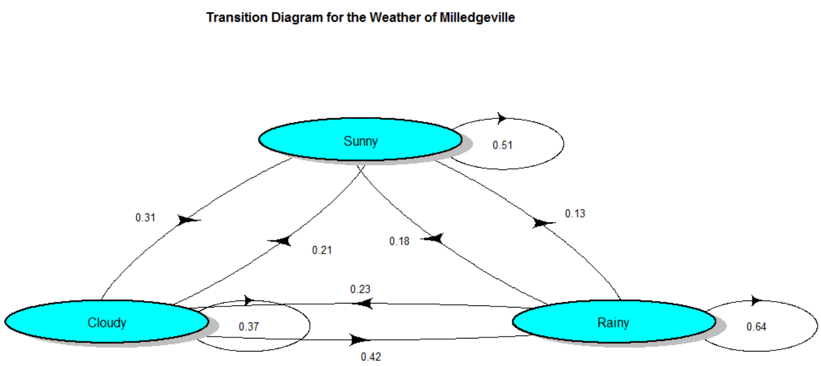
3.4 Estimating the Transition Probability Matrix for the Weather of Milledgeville

Using **R** we were able to estimate the transition probability matrix for the weather of Milledgeville using bootstrapping with $nboot = 20000$

$$\begin{array}{c} \\ S \\ C \\ R \end{array} \begin{array}{c} \begin{array}{ccc} S & C & R \end{array} \\ \begin{bmatrix} 0.51 & 0.31 & 0.18 \\ 0.21 & 0.37 & 0.42 \\ 0.13 & 0.23 & 0.64 \end{bmatrix} \end{array}$$

From this point, we used our estimate to show that the weather of Milledgeville has the condition for the Markov property, since the chi square statistic yielded a p-value of 0.97. Unfortunately, we had to assume time homogeneity, since we didn't have enough data to verify time homogeneity, but since weather has been noted to have the time homogeneity property, then it isn't terrible to assume it.

We also used **R** to compile this beautiful transition probability diagram



3.5 Standard Error of the Bootstrap Estimate of the Transition Probability Matrix for the Weather of Milledgeville

We used **R** to find the standard error of our bootstrap estimate of the transition probability matrix for the weather of Milledgeville,

$$\begin{array}{c} \\ S \\ C \\ R \end{array} \begin{array}{c} \begin{array}{ccc} S & C & R \end{array} \\ \begin{bmatrix} 0.0003 & 0.0003 & 0.0003 \\ 0.0002 & 0.0003 & 0.0003 \\ 0.0004 & 0.0003 & 0.0004 \end{bmatrix} \end{array}$$

which is very nice, since we know that it is very unlikely that the transition probabilities will exceed three standard deviations beyond the mean. Furthermore, we can use the SE (standard error matrix) to calculate confidence intervals, e.g., the 95% confidence interval for the true mean of all the transition probabilities for the weather of Milledgeville is determined by $\hat{P} \pm 1.96 \cdot \mathrm{SE}$, where $\hat{P}$ is our bootstrap estimate for the transition probability matrix for the weather of Milledgeville. Since all the elements of SE is considerably small, i.e., each element is less than or equal to 0.0004, where $\hat{P}$ is our bootstrap estimate for the transition probability matrix for the weather of Milledgeville and SE is the standard error matrix for our bootstrap estimate of our transition probability matrix for the weather of Milledgeville, then $\hat{P} \pm 1.96 \cdot \mathrm{SE} \approx \hat{P}$, i.e.,

$$\begin{bmatrix} \mu_{p_{SS}} & \mu_{p_{SC}} & \mu_{p_{SR}} \\ \mu_{p_{CS}} & \mu_{p_{CC}} & \mu_{p_{CR}} \\ \mu_{p_{RS}} & \mu_{p_{RC}} & \mu_{p_{RR}} \end{bmatrix} \approx \begin{bmatrix} 0.5023798 & 0.3175885 & 0.1817159 \\ 0.2065076 & 0.3721489 & 0.4229150 \\ 0.1323117 & 0.2276287 & 0.6411740 \end{bmatrix} \approx \hat{P}$$

In particular, since the standard error for sunny to sunny is 0.0003, then we're

95% confident that the true mean of the transition probability for sunny to sunny is approximately 0.64.

3.6 Projections for the Weather of Milledgeville for 2019

Now that we've established a good estimate of the transition probability matrix for Real Madrid and it has the properties of a regular Markov chain, we can use the n-transition probability theorem to predict what the weather of Milledgeville will be in the future. The weather of Milledgeville for 12/31/18 was rainy, so let's consider $P[X_{50} = R|X_0 = R]$. Using the n-step transition probability theorem,

$$\begin{aligned} S_{50} &= S_0 \cdot P^{50} \\ & \qquad \begin{matrix} S & C & R \end{matrix} \\ &\approx \begin{bmatrix} 0.25 & 0.29 & 0.46 \end{bmatrix} \\ P[X_{50} = R|X_0 = R] &\approx 0.46 \end{aligned}$$

Furthermore, we notice $S_{100} = \begin{bmatrix} 0.25 & 0.29 & 0.46 \end{bmatrix}$ which is S_{50}, so $\begin{bmatrix} 0.25 & 0.29 & 0.46 \end{bmatrix}$ is the stationary distribution, so by multiplying the stationary distribution by the total number of days in 2019, which is 365 days, the weather of Milledgeville is projected to have 90 sunny days, 106 cloudy days, and 169 rainy days for 2019...

4 Conclusions

Many times throughout mine and Dr. Jebessa Mijena's research on the Markov process we discussed how someone could use the information that we came up with about Real Madrid for the current season and the weather of Milledgeville for the year 2019. Two of the main ideas that we discussed is that a casino could very well use the information on Real Madrid to help with making odds, and as for the weather for Milledgeville for the year of 2019 someone could use what we came up with for agriculture, construction, or anything that pertains to working outside.

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