The Comparison of the Trapezoid Rule and the Gaussian Quadrature

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1 Abstract

This paper provides a detailed exploration of the comparison of the Gaussian Quadrature and the Trapezoid Rule. They are both numerical methods that are used to approximate integrals. The Gaussian Quadrature is a much better approximation method than the Trapezoid Rule. We will prove and demonstrate the approximating power of both rules, and provide some practical applications of the rules as well.

2 Introduction

Integration is the calculation of the area under the curve a function $f(x)$. **The integral of** f produces a family of antiderivatives $F(x)+c$ that is used to calculate the area, denoted by,

$$\int f(x)dx.$$

A definite integral calculates the area under the curve of a function $f(x)$ on a specific interval $[a, b]$, denoted by,

$$\int_a^b f(x)dx.$$

The Fundamental Theorem of Calculus is how a definite integral should be calculated; the theorem states the following:

$$\int_a^b f(x)dx = F(b) - F(a).$$

There are many methods developed to find $F(x)$ that will produce the exact area. These methods include integration by parts, partial fraction decomposition, change of variables, and trig substitution. However, there are some functions that can not be integrated exactly. They must be approximated. We will discuss two approximation methods in numerical analysis called the Trapezoid Rule and the Gaussian Quadrature. The definition of a Quadrature rule for nodes

$x_1 < x_2 < x - 3 < \cdots < x_n$ is

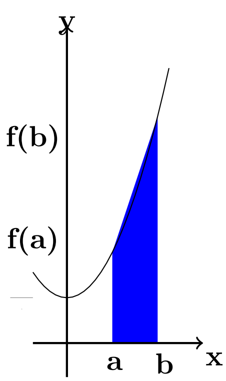
$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i).$$

A **closed quadrature** rule means that $f(x)$ is evaluated at the endpoints. An **open quadrature** rule is when $f(x)$ is not evaluated at the endpoints.

3 Trapezoid Rule

The **Trapezoid Rule** is a closed Quadrature rule where $f(x)$ is only evaluated at the endpoints a and b on the interval $[a, b]$. Visually on a graph, we see the line that passes through points $(a, f(a))$ and $(b, f(b))$, which forms a trapezoid with the x-axis. Thus, the area under the curve can be approximated by the area a Trapezoid, as shown in Figure 1.

Figure 1



The area of a Trapezoid is,

$$\int_a^b f(x)dx = \frac{f(a) + f(b)}{2}(b - a).$$

Since the Trapezoid Rule is a Quadrature rule will format the area of the trapezoid so that it matches the definition. Note the following:

$$\begin{aligned}
\int_a^b f(x)dx &= \frac{f(a) + f(b)}{2}(b - a) \\
&= \frac{(b - a)}{2}(f(a) + f(b)) \\
&= \frac{(b - a)}{2} f(a) + \frac{(b - a)}{2} f(b) \\
&= c_1 f(a) + c_2 f(b) \\
&= \sum_{i=1}^{2} c_i f(x_i).
\end{aligned}$$

The area of a Trapezoid is now in the form of a Quadrature Rule with $n = 2$. Note that $c_1 = c_2 = \frac{(b-a)}{2}$. That is, the coefficients are always that same. Since the Trapezoid Rule takes the area under a linear function, the value of the summation is exact only for polynomials of degree ≤ 1. For every polynomial of degree > 1, the Trapezoid Rule will over- or under- approximate. Consider the Lagrange Polynomial interpolation for 2 points $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

$$P_1(x) = f(x_0)\frac{(x - x_1)}{x_0 - x_1} + f(x_1)\frac{(x - x_0)}{x_1 - x_0} + Of''(c_x)$$

where $Of''(c_x)$ is the error term from the approximation and c_x depends continuously on x_0 and x_1. For a linear function $f(x) = kx + C$ for some constants k and C, the second derivative $f''(x) = 0$. Therefore, whenever $f(x)$ is a linear function, the error will be zero and thus the approximation will be exact. In the next couple of sections, we will see how the coefficients and nodes of the Gaussian Quadrature are specially chosen to produce high accuracy for higher degree polynomials. First, we will explore some underlying information about the Legendre Polynomials.

4 The Gaussian Quadrature

4.1 Legendre Polynomials

Before we learn about the Gaussian Quadrature, we need to understand the Legendre Polynomials. The **Legendre Polynomials** are a set of nonzero polynomials $S = \{P_1(x), P_2(x), P_3(x) \dots\}$ that are orthogonal on the interval $[-1, 1]$. By the definition of orthogonal,

$$\int_{-1}^{1} P_i(x)P_j(x)dx = \begin{cases} = 0 & \text{if } i \neq j \\ \neq 0 & \text{if } i = j. \end{cases}$$

The next Theorem will help us understand why the property of orthogonal functions is valuable for the Gaussian Quadrature.

Theorem (1): If $\{p_1, p_2, ..., p_n\}$ is a set of nonzero orthogonal polynomials on the closed interval $[a, b]$ where the degree of every $p_i = i$ for $i \in \{1, 2, ..., n\}$, then every p_i has i distinct roots on the interval (a, b).

By Theorem (1), every $P_i(x) \in S$ has i distinct roots on $[-1, 1]$. We will refer back to these roots in the following section.

4.2 Gaussian Quadrature

Recall that the Trapezoid Rule is exactly accurate for polynomials of degree ≤ 1. Suppose we wanted to develop a Quadrature rule that is accurate for polynomials of a much higher degree. Hence, we want a linear combination of values $f(x_i^*)$ that is either exactly or very close to the actual value of the integral. Say that we want to integrate a polynomial function $P_i(x)$ on $[-1, 1]$ where $n = 2$.

$$\int_{-1}^{1} P(x)dx = \sum_{i=1}^{2} c_i P(x_i) = c_1 P(x_1) + c_2 P(x_2).$$

Consider the integrals of the first four polynomials.

$$\int_{-1}^{1} 1dx = c_1 + c_2,$$

$$\int_{-1}^{1} xdx = c_1(x_1) + c_2(x_2),$$

$$\int_{-1}^{1} x^2dx = c_1(x_1)^2 + c_2(x_2)^2,$$

$$\int_{-1}^{1} x^3dx = c_1(x_1)^3 + c_2(x_2)^3.$$

Since we have four unknowns c_1, c_2, x_1, and x_2, and 4 equations, we have a solvable system. When this system is solved, we obtain $c_1 = c_2 = 1$, $x_1 = \sqrt{\frac{1}{3}}$ and $x_2 = -\sqrt{\frac{1}{3}}$. Note that $\sqrt{\frac{1}{3}}$ and $-\sqrt{\frac{1}{3}}$ are the roots of the 2^{nd} degree Legendre polynomial. When we extend this example to the general n^{th} case, we obtain that each x_i is a root of the i^{th} Legendre polynomial.

$$\int_{-1}^{1} dx = c_1 + c_2 + \ldots + c_n = 2$$
$$\int_{-1}^{1} x dx = c_1x_1 + c_2x_2 + \ldots + c_nx_n = 0$$
$$\cdots$$
$$\int_{-1}^{1} x^{2n-2} dx = c_1x_1^{2n-2} + \ldots + c_nx_n^{2n-2} = \tfrac{1}{2n-1} - \tfrac{(-1)^{2n-1}}{2n-1}$$
$$\int_{-1}^{1} x^{2n-1} dx = c_1x_1^{2n-1} + \ldots + c_nx_n^{2n-1} = \tfrac{1}{2n} - \tfrac{(-1)^{2n}}{2n}$$

We have that the **Gaussian Quadrature** is a linear combination of the function $f(x)$ evaluated at the roots of the n^{th} Legendre polynomial. In the n^{th} case shown, we can see that the degree of the last polynomial in the series of equations is $2n-1$. Hence, for a given n, the Gaussian Quadrature is exact up to polynomials of degree $2n-1$. Clearly, this is a much more accurate method than the Trapezoid Rule. Table 1 gives the roots and coefficients up to $n = 4$.

Table 1

n	x_i	c_i
2	$\frac{1}{\sqrt{3}}$	1
	$\frac{-1}{\sqrt{3}}$	1
3	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	0	$\frac{8}{9}$
	$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
4	$-\sqrt{\frac{15+2\sqrt{30}}{35}}$	$\frac{90-5\sqrt{30}}{180}$
	$-\sqrt{\frac{15-2\sqrt{30}}{35}}$	$\frac{90+5\sqrt{30}}{180}$
	$\sqrt{\frac{15-2\sqrt{30}}{35}}$	$\frac{90+5\sqrt{30}}{180}$
	$\sqrt{\frac{15+2\sqrt{30}}{35}}$	$\frac{90-5\sqrt{30}}{180}$

5 Comparison of Applications

We will look at a simple example that will demonstrate the accuracy of the Gaussian Quadrature and the Trapezoid Rule. Consider the integral of $f(x) = x^2$ from $[-1, 1]$. The exact value of this integral can be found by the standard integration rule for polynomials.

$$\int_{-1}^{1} x^2 dx = \frac{x^3}{3}|_{-1}^{1} = \frac{1}{3} - (\frac{-1}{3}) = \frac{2}{3}.$$

The next integral is the result from using the Trapezoid Rule.

$$\int_{-1}^{1} x^2 dx = \frac{(1-(-1))}{2}((-1)^2+(1)^2) = (1)(2) = 2.$$

Now, we show the result from the Gaussian Quadrature with two points.

$$\int_{-1}^{1} x^2 dx = (\sqrt{\frac{1}{3}})^2 + (-\sqrt{\frac{1}{3}})^2 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

We see that the Gaussian Quadrature produced the exact answer, while the Trapezoid Rule was off by $\frac{4}{3}$.

Consider the integral of a function $f(x)$ on $[a, b] \neq [-1, 1]$. The integral is not given on $[-1, 1]$, and therefore the Gaussian Quadrature cannot be applied directly to it. We must use a substitution for x in order to normalize the function onto $[-1, 1]$. Let

$a = k_1t_1 + k_2$ and $b = k_1t_2 + k_2$, where t_1 and t_2 are the upper and lower bounds of the interval we want to normalized $[a, b]$ to. In this particular case, $[t_1, t_2] = [-1, 1]$; that is, $t_1 = -1$ and $t_2 = 1$. Note that adding and subtracting a and b yield the following:

$$a + b = -k_1 + k_2 + k_1 + k_2 = 2k_2,$$
$$a - b = -k_1 + k_2 - k_1 - k_2 = -2k_1.$$

Therefore, $k_1 = \frac{b-a}{2}$ and $k_2 = \frac{a+b}{2}$. Hence, the general $x = k_1t + k_2 = \frac{b-a}{2}t + \frac{a+b}{2}$. We will demonstrate this substitution an the example where $f(x) = \ln(x)$ and $[a, b] = [1, 2]$.

$$\int_1^2 ln(x)dx = \int_{-1}^1 \ln((\frac{b-a}{2}t) + (\frac{a+b}{2}))(\frac{b-a}{2})dt$$
$$= \int_{-1}^1 \ln(\frac{t+3}{2})(\frac{1}{2})dt.$$
$$= (1)\ln(\frac{\sqrt{\frac{1}{3}}+3}{2})\frac{1}{2} + (1)\ln(\frac{-\sqrt{\frac{1}{3}}+3}{2})\frac{1}{2}$$
$$\approx 0.09585 + 0.29073$$
$$\approx 0.38658.$$

This integral can be exactly solved by using integration by parts, which yeilds $\approx$ 0.38629. The error between the Gaussian Quadrature and the real result is $\approx |0.38629 - 0.38658| = 0.00029$. The error is very close to 0. The Trapezoid Rule yeilds,

$$\int_1^2 ln(x)dx = \frac{2-1}{2}(\ln(1) + \ln(2)) \approx 0.3465.$$

The error is $|0.38629 - 0.3465| = 0.0397 > 0.00029$.

Consider $f(x) = \tan^{-1}(x)$ on $[0, 1]$. The interval is half of $[-1, 1]$.

$$\int_0^1 tan^{-1}(x)dx = \int_{-1}^1 tan^{-1}(\frac{1}{2}t + \frac{1}{2})\frac{1}{2}dt$$
$$= \frac{1}{2}(tan^{-1}(\frac{1}{2}(\sqrt{\frac{1}{3}})) + \frac{1}{2}) + \frac{1}{2}(tan^{-1}(\frac{1}{2}(-\sqrt{\frac{1}{3}})) + \frac{1}{2})$$
$$= \frac{1}{2}(tan^{-1}(\frac{\sqrt{3}+3}{6})) + \frac{1}{2}(tan^{-1}(\frac{-\sqrt{3}+3}{6}))$$
$$\approx 0.4380.$$

The error is $|0.4388 - 0.4380| \approx 0.0008$.

6 Other Applications

6.1 Improper Integrals

The Gaussian Quadrature can be used on convergent improper integrals. Consider the following convergent improper integral,

$$\int_{-1}^1 \frac{e^{-\cos x^2}}{1-x^2}dx.$$

This function is not exactly integrable using standard integration methods, and thus we need to use a Quadrature rule. Note that the function $f(x) = \frac{e^{-\cos x^2}}{1-x^2}$ is undefined at $x = 1$ and $x = -1$. Therefore, a closed Quadrature Rule is not applicable to approximate this integral. The Gaussian Quadrature is an excellent choice for the approximation method due to high accuracy and no issues with the domain on which we evaluate $f(x)$ since for every x_i is such that $-1 < x_i < 1$. Table 2 shows the evaluations of the improper integral with increasing values of n. The approximations improve as $n \to \infty$.

Table 2

n	Gaussian Quadrature
2	1.0597
3	1.1868
4	1.2536
5	1.2954
6	1.3242
7	1.3453
8	1.4364
9	1.3739
10	1.3842
11	1.3926
12	1.3997

6.2 Probability Density

A practical application of the Gaussian Quadrature is to use it to approximate the value of probability density, ρ. Probability density is represented by the following integral:

$$P(a \le x \le b) = \int_a^b \rho dx = \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

For a given interval $[a, b]$, the density ρ gives the probability that x will be between a and b. The standard normal distribution is when the mean $\mu = 0$ and the standard deviation $\sigma = 1$,

$$\int_a^b \rho dx = \frac{1}{\sqrt{2\pi}} \int_a^b e^{\frac{-x^2}{2}} dx.$$

Figure 2 shows the standard normal distribution.

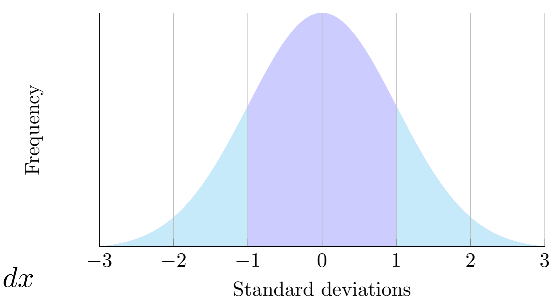


Figure 2

Table 3 gives the approximation values for $P(-1 < x < 1)$ on the standard distribution. The standard distribution value from the table for $[-1, 1]$ is $P(-1 < x < 1) \approx 0.6827$.

$$P(-1 \le x \le 1) = \int_{-1}^1 \rho dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 e^{\frac{-x^2}{2}} dx$$

Table 3

n	Gaussian Quadrature
2	0.6754
3	0.6830
4	0.6827
5	0.6827
6	0.6827

As shown in Table 3, the approximation is exactly what was desired. Table 4 shows the approximated values for three other intervals, the integral for which we needed the substitution,

$$\int_a^b \rho dx = \int_{-1}^{1} e^{\frac{-(\frac{b-a}{2}t+\frac{a+b}{2})^2}{2}} (\frac{b-a}{2})dt.$$

The following are values of density from the standard distribution table:

$$P(-2 \leq x \leq -3) \approx 0.0215$$
$$P(1 \leq x \leq 3) \approx 0.0.1574$$
$$P(0 \leq x \leq 2) \approx 0.47725$$

We now display Table 4, showing values obtained from the Gaussian Quadrature.

Table 4

n	(-2,-3)	(1,3)	(0,2)
2	0.0214	0.1594	0.4798
3	0.0214	0.1573	0.4771
4	0.0215	0.1573	0.4773

7 Conclusion

Our detailed exploration of the Trapezoid Rule and the Gaussian Quadrature has explicitly exhibited the high approximating power of the Gaussian Quadrature verses the Trapezoid Rule. Integrals show up everywhere in mathematics, physics, engineering, economics, and any other math-related field. This method is a very good one to keep in the back pocket of many mathematical projects.

8 References

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