The Helix Conjecture

A Study of Fractional Derivatives of Sinusoidal Functions

by Means of Fourier Transform

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Abstract

Students who take a calculus class learn about the first and second derivatives as well as their properties but what about the fractional derivative or even the functional derivative? From this idea stems Fractional Calculus. In this research, we will consider one definition for the fractional derivative by means of Fourier Transform. More specifically we will study the properties and applications of the fractional derivatives by means of Fourier Transform of sinusoidal functions as well as their graphical interpretations. From this investigation, we developed a visual depiction of the fractional derivative called the Helix Conjecture. The Helix Conjecture attempts to describe the movement of a sinusoidal function though the real and the imaginary plane as one takes consecutive half-derivatives. One can visualize this movement through the real and imaginary planes as a helix translating and rotating through space.

1 Introduction

Derivatives act as the backbone of Calculus and have many real life applications. One can use derivatives to solve for the rate of change, optimize a function, linear approximate a function, approximate solutions to equations and much more. Fractional Calculus is a branch of mathematics that studies different ways of treating the derivative as an operator and calculating the effects of real and complex powers. The goal of this research is to gain a better understanding of the many different definitions of fraction-ordered derivatives and their applications. We will be building upon previous research in fractional calculus done at Georgia College using the fractional derivative by means of Fourier transform. More specifically we will be working with the operator $J^{\alpha m}$ that was previously discussed in the research of Mathew Pearson and Dr. Yue in order to demonstrate if some of the basic properties of normal derivatives still hold true for fraction-ordered derivatives. We will also examine the graphical properties of the half derivatives of sinusoidal functions. In order to understand the two definitions, one must first have a basic understanding of Fourier series and Fourier transformations. Both of which will be discussed further in this paper.

2 Mathematical Background

2.1 Fourier Series

A Fourier series is an expansion of a periodic function in terms of an infinite sum of sines and cosines. Fourier series make use of the orthogonality relationships of the sine and cosine functions. The computation and study of Fourier series is known as harmonic analysis and is extremely useful as a way to break up an arbitrary periodic function into a set of simple terms that can be plugged in, solved individually, and then recombined to obtain the solution to the original problem or an approximation to it to whatever accuracy is desired or practical. The formula of a Fourier series can be found below.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cos(nx) + \sum_{n=1}^{\infty} b_n sin(nx) \tag{1}$$

Where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx)dx \end{aligned}$$

Also note that $f(x)$ is a piecewise continuous periodic function of period 2π. After one grasps the general idea of Fourier series they can then move onto Fourier transformations.

2.2 Fourier Transform

The Fourier Series showed us how to rewrite any periodic function into a sum of sines and cosines. The Fourier Transform is the extension of this idea to non-periodic functions. The general definition of a Fourier transform is

$$F(k) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \tag{2}$$

With the general definition of an Inverse Fourier Transform being

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{ikt}dt \tag{3}$$

However for the sake of this research we chose to maintain symmetry between the definition of a Fourier Transform and its inverse. Since $\frac{1}{2\pi}$ is a scalar it can be multiplied to either the Fourier Transform or its inverse as long as one maintains a consistent definition throughout their work. So to maintain symmetry between the two definitions we used a scalar of $\frac{1}{\sqrt{2\pi}}$ in front of both equations to get

Fourier Transform

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-ikt}dt \tag{4}$$

Inverse Fourier Transform

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikt}dt \tag{5}$$

However for common functions such as sin(x) and cos(x) the Fourier transform can be looked up on a table.

Below is a table of common Fourier Transform pairs that will be used in this research

Function, $f(t)$	Fourier Transform, $F(k)$
$sin(k_0 t)$	$i\sqrt{\frac{\pi}{2}}[\delta(k-k_0)+\delta(k+k_0)]$
$cos(k_0 t)$	$\sqrt{\frac{\pi}{2}}[\delta(k-k_0)-\delta(k+k_0)]$
e^{ikt}	$\sqrt{2\pi}\delta(k-k_0)$
$f(t-t_0)$	$F(k)e^{-ikt}$

This research has an emphasis on sine and cosine functions so the Fourier Transform of each is listed below.

$$\hat{sin(kt)} = \frac{\pi}{i\sqrt{2\pi}}[\delta(k_0+k)-\delta(k_0-k)]$$
$$\hat{cos(kt)} = \frac{\pi}{\sqrt{2\pi}}[\delta(k_0-k)+\delta(k_0+k)]$$

Notice that many of these Fourier transform pairs involve a δ. This is Dirac's Delta function. The delta function has a special property that is very useful for Fourier transforms.

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) \tag{6}$$

2.3 Dirac's Delta Function

The delta function is a special function with a property that is very useful for Fourier transforms.

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) \tag{7}$$

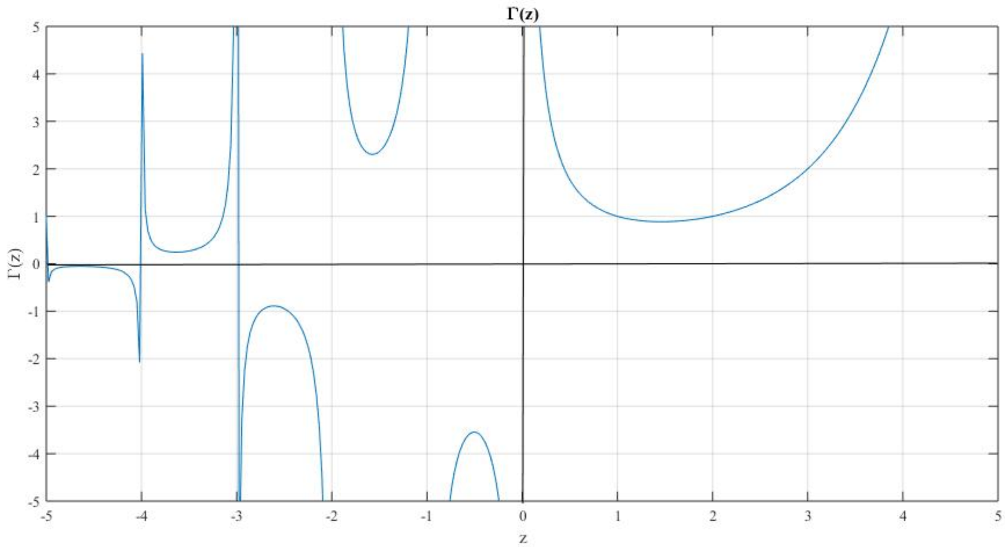

Figure 1: Gamma Function Approximation

2.4 The Gamma Function

The Gamma function is the generalization of the factorial for all real numbers.

$$\Gamma(z) = \int_0^\infty e^{-u}u^{z-1}du, \text{ for all } z \in \mathbb{R} \tag{8}$$

By using integration by parts on the Gamma Function we find the property that will be of most importance to this research

$$\Gamma(z+1) = z\Gamma(z), \text{ when } z \in \mathbb{N}_+, \Gamma(z) = (z-1)! \tag{9}$$

We can use the Gamma Function to get an approximate value of a non integer factorial

For example

$$(0.5)! \approx \int_0^\infty e^{-u}u^{0.5-1}du \approx 1.777245385 \tag{10}$$

2.5 The Riemann-Liouville Approach

Derived from Cauchy's formula for the n^{th} integration of a function, the Riemann-Louiville approach to fractional derivatives is one of the most common definitions for a fractional derivative.

Below is Cauchy formula for the n^{th} derivative of a function where $n \in \mathbb{Z}^+$

$$D^n f(t) := f_n(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau \tag{11}$$

But what if we were to replace n with a non integer? Since $\Gamma(\alpha) = (n-1)!$ we can replace it in Cauchy's general equation.

Thus the Riemann-Louiville Approach was born Below is a simple example of a fractional derivative using the Riemann-Louiville Approach.

$$\begin{aligned}(x)^{(\frac{1}{2})} &= \frac{1}{\Gamma(\frac{1}{2})} \int_0^x (x-\tau)^{-\frac{1}{2}} \tau d\tau \\ &\approxeq \frac{1}{1.777245} \cdot \frac{4}{3} x^{\frac{3}{4}}\end{aligned}$$

3 Fractional Derivative by Means of Fourier Transform

From previous research at Georgia College [3] this formula for function-ordered derivatives was derived. This formula was then confirmed through further reading on the subject of fractional derivatives.

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} (ik)^\alpha \hat{f}(k) dk \tag{12}$$

Where $\hat{f}(k)$ is the fourier transform of $f(x)$. From this formula we are able to find the fractional derivatives of periodic functions such as $sin(x)$ and $cos(x)$. The work for the half derivatives of the two previously mentioned functions can be found below.

Half derivative of $sin(x)$

$$\begin{aligned}
[sin(x)]^{(\frac{1}{2})} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}(ik)^{\frac{1}{2}}\hat{sin(x)}(k)dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(ik)^{\frac{1}{2}}\frac{\pi}{i\sqrt{2\pi}}[\delta(k-1)-\delta(k+1)]e^{ikx}dk \\
&= \frac{1}{2\sqrt{i}}\int_{-\infty}^{\infty}k^{\frac{1}{2}}\delta(k-1)e^{ikx}dk - \int_{-\infty}^{\infty}k^{\frac{1}{2}}\delta(k+1)e^{ikx}dk \\
&= \frac{1}{2\sqrt{i}}[e^{ix}-ie^{-ix}] \\
&= i\cdot sin(x-\frac{\pi}{4})
\end{aligned}$$

The half derivative of $cos(x)$

$$\begin{aligned}
[cos(x)]^{(\frac{1}{2})} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}(ik)^{\frac{1}{2}}\hat{cos(x)}(k)dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(ik)^{\frac{1}{2}}\frac{\pi}{\sqrt{2\pi}}[\delta(k-1)+\delta(k+1)]e^{ikx}dk \\
&= \frac{\sqrt{i}}{2}\int_{-\infty}^{\infty}k^{\frac{1}{2}}\delta(k-1)e^{ikx}dk + \int_{-\infty}^{\infty}k^{\frac{1}{2}}\delta(k+1)e^{ikx}dk \\
&= \frac{\sqrt{i}}{2}[e^{ix}+ie^{-ix}] \\
&= i\cdot sin(x+\frac{\pi}{4}) \\
&= i\cdot cos(x-\frac{\pi}{4})
\end{aligned}$$

Now we can graph this result using MATLAB. Note that the result is entirely imaginary so the vertical axis on the graph is imaginary.

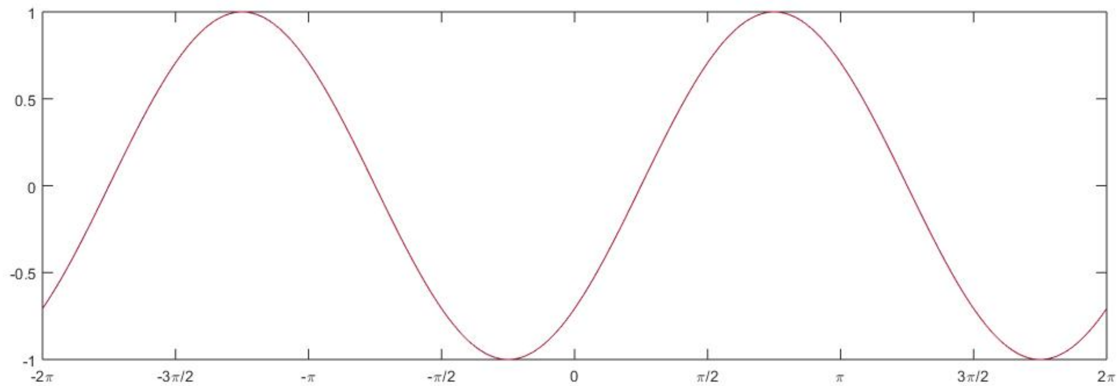

Figure 2: Half Derivative of sin(x)

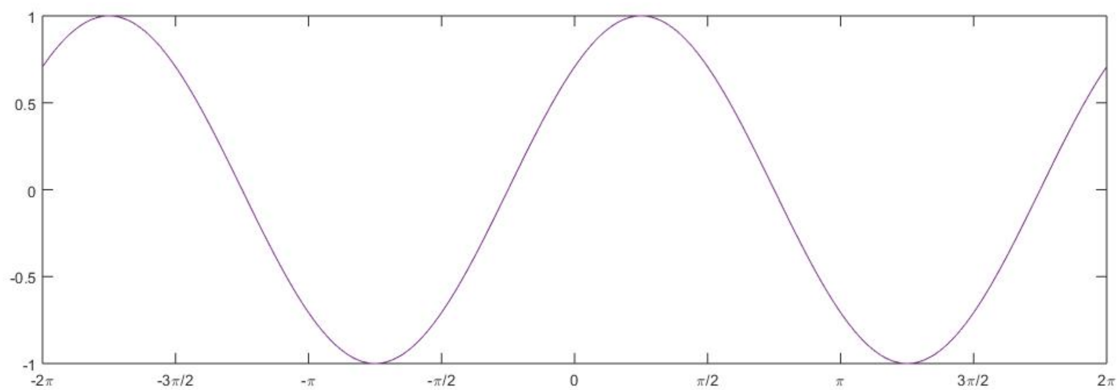

Figure 3: Half Derivative of cos(x)

When comparing the two results we noticed something rather interesting. The half derivative of both the sine and cosine functions resulted in an entirely imaginary version of the original function phase shifted to the right $\frac{\pi}{4}$. If this pattern were to hold true that would mean if we took the half derivative of the respective functions once more, we would return to the real plane and phase shift to the right $\frac{\pi}{4}$. The resulting functions of taking the half derivative twice would then be consistent with the first derivatives of sine and cosine. This should also be true since we are treating the derivative as an operator D so $(D^{(\frac{1}{2})})^{(\frac{1}{2})} = D^{(1)}$ should also be true. Now we can work through this mathematically to see if this is true.

The half derivative of half derivative of $sin(x)$

$$
\begin{aligned}
[[sin(x)]^{(\frac{1}{2})}]^{(\frac{1}{2})} &= i \cdot sin(x - \frac{\pi}{4}) \\
&= i \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}(ik)^{(\frac{1}{2})} sin(\hat{x} - \frac{\pi}{4}) dk \\
&= \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}(ik)^{(\frac{1}{2})} \hat{sin(x)} e^{i\frac{\pi}{4}k} dk \\
&= \frac{\pi}{\sqrt{2\pi}} \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x-\frac{\pi}{4})}(ik)^{(\frac{1}{2})} [\delta(k-1) - \delta(k+1)] dk \\
&= \frac{\sqrt{i}}{2} [e^{ik(x-\frac{\pi}{4})} - e^{-i(x-\frac{\pi}{4})}] \\
&= sin(\frac{\pi}{2} - x) \\
&= cos(x) \checkmark
\end{aligned}
$$

The half derivative of half derivative of $cos(x)$

$$
\begin{aligned}
[[cos(x)]^{(\frac{1}{2})}]^{(\frac{1}{2})} &= i \cdot cos(x - \frac{\pi}{4}) \\
&= i \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}(ik)^{(\frac{1}{2})} cos(\hat{x} - \frac{\pi}{4}) dk \\
&= \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}(ik)^{(\frac{1}{2})} \hat{cos(x)} e^{i\frac{\pi}{4}k} dk \\
&= \frac{\pi}{\sqrt{2\pi}} \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x-\frac{\pi}{4})}(ik)^{(\frac{1}{2})} [\delta(k-1) + \delta(k+1)] dk \\
&= \frac{i}{2} [e^{ik(x-\frac{\pi}{4})} + e^{-i(x-\frac{\pi}{4})}] \\
&= cos(\frac{\pi}{2} - x) \\
&= -sin(x) \checkmark
\end{aligned}
$$

Therefore the results are consistent with the operator definition of the derivatives as well as consistent with our graphical observations of these functions. From these intriguing results stems the heart of this research as we attempt to package these findings into a concise mathematical conjecture.

Also note that these results are consistent with the general equation for a fractional derivative that was derived in previous research at Georgia College and State University.

$$\frac{d^{\frac{m}{n}}}{dx^{\frac{m}{n}}} sin(\omega x) = \frac{i^{\frac{m}{n}} \omega^{\frac{m}{n}}}{2i}(e^{i\omega x} - (-1)^{\frac{m}{n}} e^{-i\omega x}) \tag{13}$$

4 The Helix Conjecture

The Helix Conjecture stems from our observations above and states that the half derivative of a sinusoidal function shifts the real values to imaginary and vice versa while phase shifting the graph to the right $\frac{\pi}{4}$. One can visualize this movement through the real and imaginary planes as a helix translating and rotating through space. Though a formal proof of the Helix Conjecture has not been completed, we can further see the graphical effects of the one half derivative and the helix conjecture when comparing the one fourth derivative and the three fourths derivative of $sin(x)$. Since when we treat the derivative as an operator, $[D^{(\frac{1}{4})}]^{(\frac{1}{2})} = D^{(\frac{3}{4})}]$.

Using the general equation for a fractional derivative of $sin(x)$ (equation 8) we can find the one fourth and three fourths derivatives of of $sin(x)$.

One fourth derivative of $sin(x)$

$$\begin{aligned} \frac{d^{\frac{1}{4}}}{dx^{\frac{1}{4}}} sin(x) &= \frac{i^{\frac{1}{4}}}{2i}(e^{ix} - (-1)^{\frac{1}{4}} e^{-ix}) \\ &= \frac{1}{2i^{\frac{3}{4}}}(e^{ix} - (-1)^{\frac{1}{4}} e^{-ix}) \end{aligned}$$

Three fourths derivative of $sin(x)$

$$\begin{aligned} \frac{d^{\frac{3}{4}}}{dx^{\frac{3}{4}}} sin(x) &= = \frac{i^{\frac{3}{4}}}{2i}(e^{ix} - (-1)^{\frac{3}{4}} e^{-ix}) \\ &= \frac{1}{2i^{\frac{1}{4}}}(e^{ix} - (-1)^{\frac{3}{4}} e^{-ix}) \end{aligned}$$

It is difficult to see in the current form of the results above but now the derivatives have both imaginary and real values so we will have to two graphs per derivative.

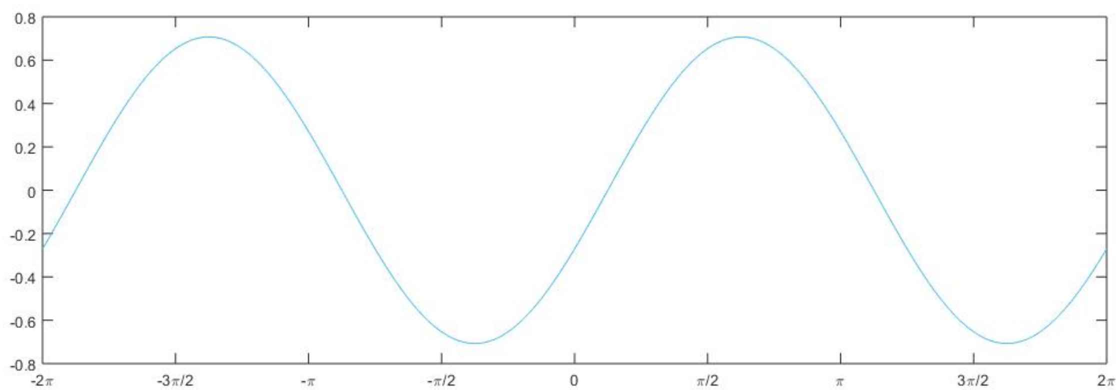

Figure 4: Real One Fourth Derivative of sin(x)

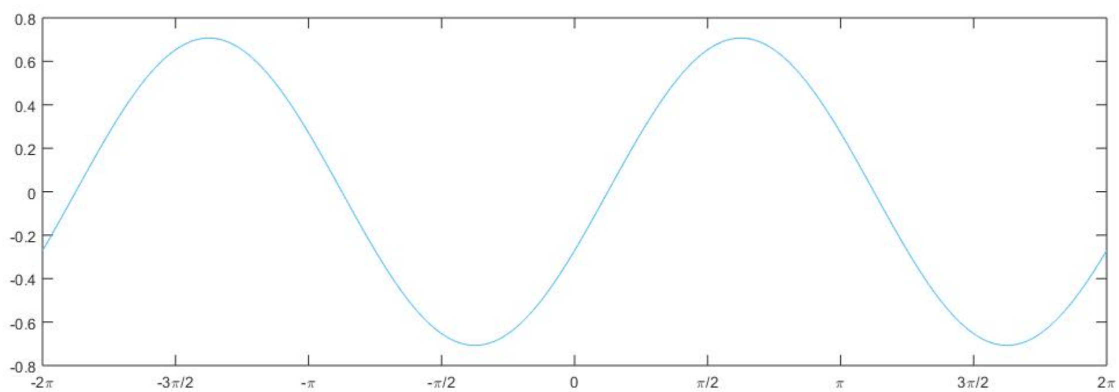

Figure 5: Imaginary One Fourth Derivative of sin(x)

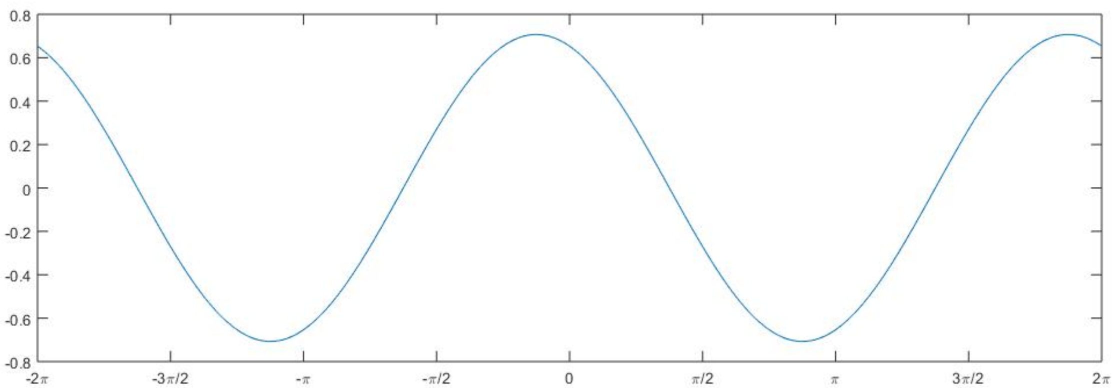

Figure 6: Real Three Fourth Derivative of sin(x)

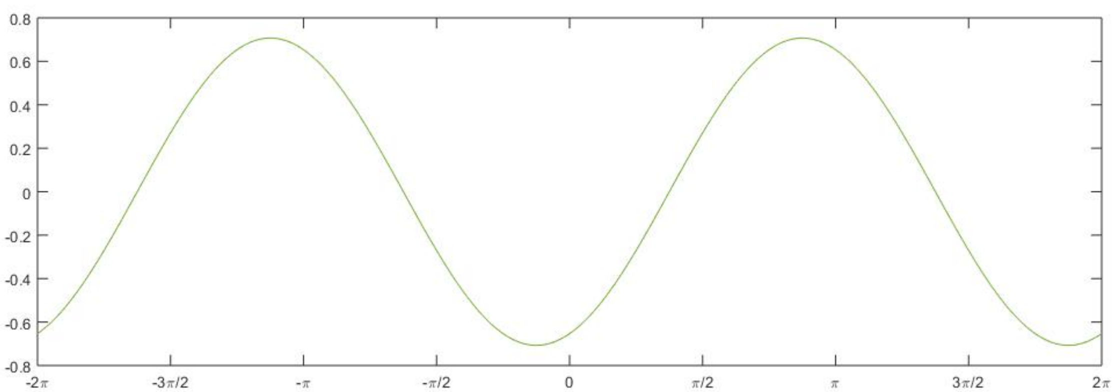

Figure 7: Imaginary Three Fourth Derivative of sin(x)

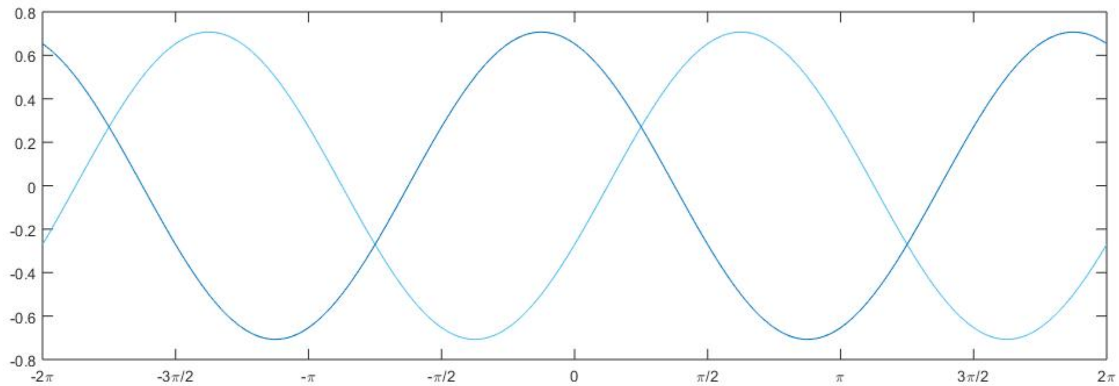

Figure 8: Real One Fourth Derivative of sin(x) and Imaginary Three Fourths Derivative of sin(x)

Now we can observe the graphs and see that the Helix Conjecture still holds true from the real one fourth derivative of $sin(x)$ to the imaginary three fourths derivative of $sin(x)$. Which can be seen even more clearly above as the graphs shits to the right $\frac{\pi}{2}$ Note that the darker blue graph is the one fourth derivative and the lighter blue graph is the three fourths derivative.

The Helix Conjecture is an interesting graphical observation of the effects of the fractional derivatives of sinusoidal functions. Through the graphical observations we can guess that the Fourier-Fractional Derivative operator is in part a translation operator as well as a rotational operator where it rotates these functions through the complex plane.

5 The Product Rule of Fractional Derivatives

Through further exploration we attempted to see if the product rule for derivatives worked for fractional derivatives. However after multiple attempts not working we remembered that the product rule only works for the first derivative and not on any higher ordered derivatives so it makes sense that it does not work for the half derivative. Instead we went about trying to demonstrate the nature of half derivatives using properties of fractional exponents and the product rule for derivatives.

The product rule of derivatives states given two functions $f(x)$ and $g(x)$ the derivative of $f(x) \cdot g(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

For the first derivative of $sin(x)cos(x)$

$$(sin(x)cos(x))' = cos^2(x) - sin^2(x)$$

Using this rule and the property showed in the previous slides we set up the equation

$$[[sin(x)cos(x)]^{(\frac{1}{2})}]^{(\frac{1}{2})}] = cos^2(x) - sin^2(x) = cos(2x)$$

Since we treat the derivative operator we can take the negative one half derivative of each side to get

$$[sin(x)cos(x)]^{(\frac{1}{2})} = [cos(2x)]^{(-\frac{1}{2})}$$

First lets solve the left side of the equation

$$\begin{aligned}
[sin(x)cos(x)]^{(\frac{1}{2})} &= [\frac{1}{2}sin(2x)]^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}(ik)^{\frac{1}{2}}(\frac{1}{2}\hat{sin}(2x))dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(ik)^{\frac{1}{2}}\frac{\pi}{2i\sqrt{2\pi}}[\delta(k-2)-\delta(k+2)]e^{ikx}dk \\
&= \frac{1}{4\sqrt{i}}[\sqrt{2}e^{2ix} - i\sqrt{2}e^{-2ix}] \\
&= \frac{\sqrt{2}}{4\sqrt{i}}[e^{2ix} - ie^{-2kx}]
\end{aligned}$$

Now let solve the right side of the equation

$$\begin{aligned}
[cos(2x)]^{(\frac{-1}{2})} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}(ik)^{\frac{-1}{2}}\hat{cos(2x)}dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(ik)^{\frac{-1}{2}}\frac{\pi}{\sqrt{2\pi}}[\delta(k-2)+\delta(k+2)]e^{ikx}dk \\
&= \frac{1}{2\sqrt{i}}\int_{-\infty}^{\infty} k^{\frac{-1}{2}}\delta(k-2)e^{ikx}dk + \int_{-\infty}^{\infty} k^{\frac{-1}{2}}\delta(k+2)e^{ikx}dk \\
&= \frac{1}{2\sqrt{i}}[\frac{1}{\sqrt{2}}e^{2ix} + \frac{1}{i\sqrt{2}}e^{-2ix}] \\
&= \frac{1}{2\sqrt{2i}}[e^{2ix} - ie^{-2ix}]
\end{aligned}$$

Now we are left with

$$\frac{\sqrt{2}}{4\sqrt{i}}[e^{2ix} - ie^{-2kx}] = \frac{1}{2\sqrt{2i}}[e^{2ix} - ie^{-2ix}]$$

We can then rewrite 4 as $\sqrt{2}^4$ thus

$$[sin(x)cos(x)]^{(\frac{1}{2})} = \frac{1}{2\sqrt{2i}}[e^{2ix} - ie^{-2ix}] = [cos(2x)]^{(\frac{-1}{2})}\checkmark$$

Although the product rule does not hold true for half derivatives. This result is consistent with that of the product rule of the first derivative of sin(x)cos(x). We have considered looking into a more general equation for the product rule of half derivatives but due to time constraints focused more on other aspects of the Fourier-Fractional Derivative.

6 Open Questions

The Helix Conjecture along with showing the product rule is consistent for the first derivative of sin(x)cos(x), are the pinnacle of this research thus far. Given more time it would be interesting to further look into a formal proof of the Helix Conjecture.Its also intriguing to speculate if the Helix Conjecture holds true for other functions besides sinusoidal functions. Since we came to the conjecture through Fourier Functional derivative and all integrable functions can be approximated with a Fourier transformation. We can speculate that since a Fourier series is a combination of sines and cosines both of which hold true to the Helix Conjecture we can overlay this onto any function that they may too hold true to the helix conjecture. It would also be interesting to look further into finding a general definition for the product rule of half derivatives. Using the definition of a Fourier-Fractional Derivative and some of the properties discussed in this research it would also be interesting to look further into the fractional derivatives of polynomials in order to find a general form as well. With the study of fractional derivatives still being a relatively under studied area of mathematics, there are countless more directions study further into the properties and workings of fractional derivatives.

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