Music Through a Mathematical Lens

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Abstract

One thing most everyone agrees on is that music is an important part of our lives. Music has the ability to change our mood and emotions, and it is natural to create music. When individuals create something, we then want to understand it. Most people might not be aware of the occurrences of mathematics in music such as those between music and abstract algebra. In particular, certain structural choices found in music so happen to exhibit group theoretic connections. While these compositional choices might not always be made for purely mathematical reasons, it is interesting to see what algebraic structures might be found within a given piece of music. In this project, we will outline some of the operations that transform music, listen to some of these transformations, and also demonstrate the associated combinatorics of such structures as we will discover how to view music through a particular mathematical lens.

1 Introduction

This paper will explore how to view music through a mathematical lens. We will look into basic permutation operations, music theory, and how music can be explained by abstract algebra. We will analyze these structures and show how they can be explained mathematically. First, we will discuss the relevant mathematical and musical definitions that will be seen throughout this paper, and we will look at the four common operations performed on pitch-class sets including transposition, inversion, retrograde, and retrograde inversion. We will show how a piece of music can be transformed using these operations. We will also explore how two pieces of music can be transpositionally or inversionally equivalent. Finally, we will explore some of the different characteristics of pitch-class sets including cardinal numbers, even and odd intervals, and the relationship between a cardinal number and the number of associated intervals. In particular, we will discuss how these relate to forming a pitch-class set.

2 Definitions

To begin our discussion of music with a mathematical overlay, we need to introduce some mathematical and musical definitions. First, we will review some important mathematical definitions that will be relevant throughout this paper.

Definition A *permutation* of a nonempty set A is a mapping $\alpha : A \longrightarrow A$ that is both one-to-one and onto. Any group whose elements are permutations, with composition as the group operation, is called a *permutation group*.

Definition A *binomial coefficient*, denoted $\binom{n}{k}$ where n and k are integers and $0 \leq k \leq n$, is defined to be the number of distinct ways to choose a k sized subset from a set of n elements without regarding order. We can calculate the binomial coefficient with the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Definition Let n be an integer. Then two integers a and b are said to be *congruent modulo n*, denoted $a \equiv b \pmod{n}$, if and only if $n|(a-b)$.

Now we will introduce some basic musical definitions. First, note that if a note is a sound in music, then the pitch of a note means how high or low the note is. Also, an octave is the interval between one pitch and another with either half or double its frequency. Consequently, two pitches separated by an octave are of the same pitch-class. Now we will introduce the definitions of pitch-classes, and pitch-class sets.

Definition Pitches that are enharmonically equivalent or have the same pitch belong to the same *pitch-class*.

Note that B$\sharp$ and C sound exactly the same, i.e., if we were to play these notes on a piano, we would be playing the exact same key for both notes. Thus, they are enharmonically equivalent and are in the same pitch-class. We are working within a 12 pitch-class system so we can assign an integer to each pitch. This leads to the next definition.

Definition *Pitch-class integers* are the integers $\{0, ..., 11\}$ assigned to each pitch-class.

The table below from [Zha09] represents all the notes including sharps, flats, and double flats in their pitch-classes and their assigned integers. Note that there are twelve different integers just like the twelve hours on a clock. Since an octave is the interval between one note and another with half or double the first note's frequency, if we play the note C and then we play a note one octave up from C, we will be playing seven full notes higher since there are seven full notes (C,D,E,F,G,A,B) but we will be playing twelve pitches higher since we have to include all the sharps and flats. Thus, we calculate everything modulo 12 because if we play a note one octave up, we will still be playing the same note in a different pitch.

Table 1: Pitch-Class Integers

Integer	Pitches
0	B♯, C, D♭♭
1	C♯, D♭
2	D, E♭♭
3	D♯, E♭
4	E, F♭
5	E♯, F, G♭♭
6	F♯, G♭
7	G, A♭♭
8	G♯, A♭
9	A, B♭♭
10	A♯, B♭
11	B, C♭

Definition A *pitch-class set* (pc set) is a set of pitch-classes, denoted as a string of integers enclosed in brackets.

For example, the musical pitches D, F, and A♯ can be represented by the pitch-class set $[2, 5, 10]$. Note that the order doesn't change the pitches in the pitch-class set but it does change its form which will be discussed later.

Since we have a basic understanding of the notation of pitch-class sets, we will introduce the different operations we can perform on pitch-class sets as well as some examples. The operations we will explore are transposition, retrograde, inversion, and retrograde inversion.

Definition *Transposition* form moves a pitch-class or pitch-class set up by additing a constant t to each pitch class modulo 12. We say that the value t a pitch-class is moved up by is the *transposition operator*. We say that a pitch-class set B is the transposition of pitch-class set A at level t, denoted $B = T(A, t)$, where t is the transposition operator.
For example, let $A = [2, 5, 10]$ and $t = 10$. Observe the following:

$$\begin{array}{ccccccccc} \text{A} & & \text{t} & & & & \text{B} & \\ 2 & + & 10 & = & 12 & \equiv & 0 & \pmod{12} \\ 5 & + & 10 & = & 15 & \equiv & 3 & \pmod{12} \\ 10 & + & 10 & = & 20 & \equiv & 8 & \pmod{12} \end{array}$$

So we have added the transposition operator of ten to each integer in the pitch-class set and then obtained the integers $12, 15,$ and 20. However, since we are working modulo 12, we have the new pitch-class set $B = [0, 3, 8]$ and $B = T(A, 10)$.

Definition *Retrograde* form is created by writing the notes of a pitch-class set in the original version in reverse order.

For example, let A be the pitch-class set $[1, 2, 3]$. Writing this in retrograde form would give us the new pitch-class set $[3, 2, 1]$

Definiton *Inversion* form is the mapping of each pitch-class or pitch-class set to its inverse. Hence, let a' denote the inverse of a, then

$$a' = 12 - a \pmod{12}.$$

We say that a pitch-class set B is the inversion of a pitch-class set A, denoted $B = I(A)$. Note that the inverse of each pitch-class integer is as presented in the following table.

Table 2: Inverse Table

Pitch-Class Integer		Inverse
0	$\leftrightarrow$	0
1	$\leftrightarrow$	11
2	$\leftrightarrow$	10
3	$\leftrightarrow$	9
4	$\leftrightarrow$	8
5	$\leftrightarrow$	7
6	$\leftrightarrow$	6

For example, let $A = [3, 5, 11]$. We can calculate the inverse of A as follows:

$$a' = 12 - 3 \pmod{12} = 9,$$
$$a' = 12 - 5 \pmod{12} = 7, \text{ and}$$
$$a' = 12 - 11 \pmod{12} = 1.$$

Thus, the inverse of A is the new pitch-class set $B = [9, 7, 1]$.

Definition *Retrograde inversion* form is a combination of inversion and retrograde. It is the inversion form of a pitch-class set with the notes in reverse order.

For example, if we take the pitch-class set $A = [3, 5, 11]$, then from the previous example we know the inverse is $B = [9, 7, 1]$. Finally, by rearranging the pitches in reverse order, we get the retrograde inversion, $C = [1, 9, 7]$.

Definition A pitch-class set is in *normal order* if the set is in ascending numerical order at the outset and each circular permutation is kept in ascending numerical order. Circular permutations are when you take the first element in each set and place it last. In order to find the normal order of a set, one must first find all the circular permutations of the set. Then the circular permutation with the smallest difference between the first and last element is the normal order of the set. If there are two or more permutations with the same least difference, then we take the difference between the first and second integer and whichever has the least difference is the *best normal order*.

Definition A pitch-class set is in *prime form* if the first integer is zero and it is in normal order or best normal order.

Now that we have an understanding of the operations we can perform on pitch-class sets, we can now explore how two pitch-class sets can be equivalent.

Definition Two pitch-class sets are *transpositionally equivalent* if and only if they are reducible to the same prime form by transposition.

We will now look at two examples of equivalent pitch-class sets. The following two examples are referenced in [For73]. First, we can look at three measures from the musical piece Berg, Four Pieces for Clarinet and Piano Op. 5. The first measure represents pitch-class set A and the next two measures represent pitch-class set B.

A: [0, 3, 4, 7, 8, 9]

B: [4, 7, 8, 11, 0, 1]

Note that we can obtain pitch-class set B by adding the transposition operator of 4 to each pitch-class integer in pitch-class set A. Thus, these two pitch-classes are transpositionally equivalent.

Definition Two pitch-class sets are *inversionally equivalent* if and only if they are reducible to the same prime form by inversion followed by transposition.

For example, we can look as two measures from the musical piece Schoenberg, Five Pieces for Orchestra Op. 16/1. The first measure represents pitch-class set A and the second measure represents pitch-class set B.

A: [1, 4, 5, 8, 9]

B: [9, 10, 1, 2, 5]

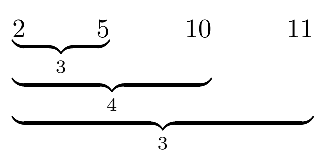
Note that we can obtain pitch-class set B by taking the inverse of each pitch-class integer in pitch-class set A and adding the transposition operator of 6 to the inverse of each pitch-class integer. Thus, these two pitch-class sets are inversionally equivalent.

3 Characteristics of Pitch-Class Sets

Now that we have an understanding of all the operations we can perform on a pitch-class set and how pitch-class sets can be equivalent, we can explore all the different ways to form pitch-class sets and the different intervals we obtain from those formations.

Definition The *cardinal number* of a pitch-class set is the cardinality of the pc set.

Note that integers in pitch-class sets form intervals. The interval formed by two pitch-class integers a and b is the difference $|a-b|$. Note that the number of intervals formed by a pitch-class set is dependent on the cardinal number of that set. We have that there are 12 intervals corresponding to the 12 pitch-classes once all the duplicates are removed. However, these 12 intervals then reduce to 6 *interval classes* by the inverse equivalence $d \equiv d' \pmod{12}$.



For example, consider the pitch-class set $A = [2, 5, 10, 11]$. To find the intervals, we look at the first integer in the pitch-class set and take the difference between that integer and the rest in the set. So we have the difference between 2 and 5 is 3, the difference between 2 and 10 is 8, but since we are working with respect to inverse equivalence modulo 12, the interval between 2 and 10 is 4, and finally the difference between 2 and 11 is 9, which is inverse equivalent to 3. We then continue this process for the rest of the integers in the set. Counting all the occurrences, there is one interval class of one, two interval classes of three, one interval class of four, one interval class of five, and one interval class of six. Thus, the total number of intervals for this particular pitch-class set is six.

If we let k denote the cardinal number and n denote the number of intervals, then we see in the table below how the cardinal number of a set connects to the number of intervals in a set.

Table 3: Cardinal Numbers and Intervals

Cardinal Number (k)	Number of Intervals n
1	0
2	$0+1$
3	$0+1+2$
4	$0+1+2+3$
$\vdots$	$\vdots$
12	$0+1+2+3+\cdots+11$

Note that cardinal number one has no intervals because there is only one element in the set so there is nothing with which to compute a difference. Cardinal number two only has one interval because there are only two elements in the set.. We can generalize this table further into the equation $n = \frac{k^2-k}{2}$.

Once we have an understanding of intervals and interval classes, we note that there are even and odd interval classes such that the odd interval classes are interval classes one, three, and five, and the even interval classes are the interval classes two, four, and six. Then, we can put all the information about cardinal numbers, number of intervals, and even and odd intervals into a table and look at connections. The table below represents the different patterns of odd and even intervals we can get from an interval with cardinal number k. Note that we do not include cardinal number 1 because that would be the equivalent of playing just one note and we can not take the interval of one note.

Table 4: Cardinality and Intervals

Cardinal Number (k)	Intervals (n)	Even Intervals	Odd Intervals
2	1	1	0
		0	1
3	3	1	2
		3	0
4	6	2	4
		3	3
		6	0
5	10	4	6
		6	4
		10	0
6	15	6	9
		7	8
		10	5
		15	0
7	21	9	12
		11	10
		15	6
8	28	12	16
		13	15
		16	12
9	36	16	20
		18	18
10	45	21	24
		20	25
11	55	25	30
12	66	30	36

Note that for every pitch-class set with cardinal number five, we will always have ten intervals and these can either be of the form four even and six odd, six even and four odd, or ten even and zero odd intervals. While determining this table, we observed that there could never be zero even and all odd intervals for any pitch-class set with cardinal number greater the two. This is because in order to form an even interval we must take the difference between either two even integers or two odd integers. Thus, for all pitch-class sets with cardinal number greater than two, we will always have at least two even or two odd integers in the set to form an even interval.

It's natural to wonder how one can determine all the even and odd intervals obtained from a pitch-class set. Studying table four, we were able to determine the relationships of these numbers using binomial coefficients. Below are the generalized tables for the number of even and odd entries in a pitch-class set and the number of intervals for each entry. Let k represent the cardinal number of a set and n denote the total number of intervals for that cardinal number. Then, we will let j be any integer such that $0 \leq j \leq k$.

Table 5: Entries and Intervals

Even	Odd	Even	Odd
0	k	n	0
1	$k-1$	$\binom{k-1}{2}+\binom{1}{2}$	$k-1$
2	$k-2$	$\binom{k-2}{2}+\binom{2}{2}$	$2(k-2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
j	$k-j$	$\binom{k-j}{2}+\binom{j}{2}$	$j(k-j)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-1$	1	$\binom{k-1}{2}+\binom{1}{2}$	$k-1$
k	0	n	0

Note that the first two columns of table five represent the possible number of even and odd entries we can have in a pitch-class set of cardinal number k. We can either have no even and all odd entries, one even and $k-1$ odd entries, ..., j even and $k-j$ odd entries, ..., all even and no odd entries. The next two columns represent the possible number of even and odd intervals we could get from these entries. If we have no even entries and all odd entries the we will have all even intervals and no odd intervals. In order to form the odd intervals on the table, we can pair each even entry with an odd entry since that is how we get an odd interval. So looking at the generalized portion of the table, whenever we have j even integers and $k-j$ odd integers, we just multiply the two together to get $j(k-j)$ odd intervals. Next looking at the generalized portion of the even intervals, we see that the first binomial coefficient represents choosing two even integers and the second represents choosing two odd integers since the way to produce and even interval is taking the difference between either two even or two odd integers. We then add the two binomial coefficients to get the total number of even intervals. In the above table it becomes apparent that some of the binomial coefficients are not defined so whenever $k-j<2$, we let $\binom{k-j}{2}=0$.

To illustrate this further, we will now look at a few examples of pitch-class sets with cardinal numbers 3, 6, and 10. Note that for a pitch-class set with cardinal number three, we can have a pc set in the form (even, even, even), (odd, odd, odd), (even, even, odd), or (even, odd, odd). Looking at these pc sets, we can see that for the sets with all even or all odd entries, we would have three even intervals and zero odd intervals. For the sets with two even or two odd entries, we would have one even interval and two odd intervals. This matches up with the information in Table 4. Using Table 5 as a guide, note the following:

Table 6: Entries and Intervals

Even	Odd	Even	Odd
0	$k = 3$	$n = 3$	0
1	$3 - 1 = 2$	$\binom{3-1}{2} + \binom{1}{2} = 1$	$3 - 1 = 2$
2	$3 - 2 = 1$	$\binom{3-2}{2} + \binom{2}{2} = 1$	$2(3 - 2) = 2$
$k = 3$	$3 - 3 = 0$	$\binom{3-3}{2} + \binom{3}{2} = 3$	$3(3 - 3) = 0$

Now we can look at cardinal number six. According to Table 4, any pc set with six entries will have fifteen intervals either of the form of 6 even and 9 odd, 7 even and 8 odd, 10 even and 5 odd, or 15 even and 0 odd. Note that a pc set of cardinal number six can be of the form [odd, odd, odd, odd, odd, odd], [odd, odd, odd, odd, odd, even], [odd, odd, odd, odd, even, even], [odd, odd, odd, even, even, even], [odd, odd, even, even, even, even], [odd, even, even, even, even, even], or [even, even, even, even, even, even].

Case 1: Take the pc set of the form [odd, odd, odd, odd, odd, odd] into consideration. Note that the only integers in the set are odd integers. Thus, when calculating the intervals we will be subtracting only odd integers. Hence, we will have only even intervals. Thus, we will have 15 even intervals and 0 odd intervals. This is the same for the pc set of the form [even, even, even, even, even, even] since the entries are all even and we will be subtracting only even integers.

Case 2: Take the pc set of the form [odd, odd, odd, odd, odd, even] into consideration. Since there is only one even entry, we pair that even with each of the odd entries to get our odd intervals. There are five odd entries so there will be five pairs of even and odd and thus five odd intervals. The rest are pairs of odd entries which will then be the ten even intervals. This will be the same for pc sets of the form [odd, even, even, even, even, even].

Case 3: Take the pc set of the form [odd, odd, odd, odd, even, even] into consideration. Since there are now two even entries, each one will need to be paired with each of the four odd entries to make an odd interval. Thus, there will be eight odd intervals. Notice there are six pairs of odd entries and one pair of even entries. Therefore, there are seven even intervals. This is the same for the pc sets of the form [odd, odd, even, even, even, even].

Case 4: Finally, take the pc set of the form [odd, odd, odd, even, even, even] into consideration. Pairing each odd entry with an even entry, we see that there are nine pairs. Thus, there are nine odd intervals. Note that there are three pairs of odds entries and three pairs of even entries. Hence, there are six even intervals.

The table on the next page agrees with these calculations.

Table 7: Entries and Intervals

Even	Odd	Even	Odd
0	$k = 6$	$n = 15$	0
1	$6 - 1 = 5$	$\binom{6-1}{2} + \binom{1}{2} = 10$	$6 - 1 = 5$
2	$6 - 2 = 4$	$\binom{6-2}{2} + \binom{2}{2} = 7$	$2(6 - 2) = 8$
3	$6 - 3 = 3$	$\binom{6-3}{2} + \binom{3}{2} = 6$	$3(6 - 3) = 9$
4	$6 - 4 = 2$	$\binom{6-4}{2} + \binom{4}{2} = 7$	$4(6 - 4) = 8$
5	$6 - 5 = 1$	$\binom{6-5}{2} + \binom{5}{2} = 10$	$5(6 - 5) = 5$
$k = 6$	$6 - 6 = 0$	$\binom{6-6}{2} + \binom{6}{2} = 15$	$6(6 - 6) = 0$

Now we will look at cardinal number ten. Note that when working with a pitch-class set with cardinal number greater than six we cannot take into account pitch-class sets with all even or all odd entries because since we are using the integers zero through 11 there are only six even and six odd. If we had a pitch-class set with cardinal number ten with all even entries then we would have to repeat entries which we cannot do. Thus, we can only look at the entries of six even and four odd, five even and five odd, and four even and six odd. Then, according to table four we have that any pitch-class set with cardinal number ten has 45 intervals either in the form of 21 even and 24 odd or 20 even and 25 odd. Table 8 shows this infomration. The other calculations are similar and are omitted from this document.

Table 8: Entries and Intervals

Even	Odd	Even	Odd
6	$10 - 6 = 4$	$\binom{10-6}{2} + \binom{6}{2} = 21$	$6(10 - 6) = 24$
5	$10 - 5 = 5$	$\binom{10-5}{2} + \binom{5}{2} = 20$	$5(10 - 5) = 25$
4	$10 - 4 = 6$	$\binom{10-4}{2} + \binom{4}{2} = 21$	$4(10 - 4) = 24$

We now understand a portion of the combinatorics associated with this music theory. Of course, composers sometimes break free from the expected in order to make their music interesting. In the end, we were able to fulfill our goal of determining some nice formulas for the even and odd intervals of cardinal number k.

References

[For73] Allen Forte. *The structure of atonal music*, volume 304. Yale University Press, 1973.

[Zha09] Ada Zhang. The framework of music theory as represented with groups. *Preprint*, 163, 2009.