Homology of Four-Dimensional Lie Algebras with A Para-Hypercomplex Structure

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1 Abstract

In this paper, we provide explicit calculations of Lie algebra homology of four-dimensional Lie algebras with a para-hypercomplex structure.

2 Introduction

The concept of Lie algebra was introduced by Marius Sophus Lie in the 1870s (see [2]), and later discovered by Wilhelm Killing (see [3]) almost a centure later. Their strong connection with Lie groups is part of the reason why they are very important in many field of physics, and is the foundation of the definition of Lie algebra cohomology. The cohomology theory for Lie algebras was first introduced in 1929 by Élie Cartan, and was later developped in 1948 by Claude Chevalley and Samuel Eilenberg. In this paper, we are interested to the notion of homology of Lie algebra which is is dual to Lie algebra cohomology. We explicitely calculate homology for four-dimensional real Lie algebras admitting an integrable, left-invariant para-hypercomplex structure using a classification provided by Blazic and al. in[1]. Since the properties of para-hypercomplex structures are not used in our calculations, we refer the reader to [1].

Recall from [4] that a Lie algebra $\mathfrak{g}$ is a vector space over a field F together with a binary operation

$$[.,.]:\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g}$$

called the Lie bracket, which satisfies the following axioms:

a) $[ax+by,z]=a\,[x,y]+b\,[y,z]\,,\quad [z,ax+by]=a\,[z,x]+b\,[z,y]$ for all scalars a, b in F and all elements x, y, z in $\mathfrak{g}$.

b) $[x,y]=-\,[y,x]$ for all elements x, y in $\mathfrak{g}$.

c) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ The Jacobi identity : for all x, y, z in $\mathfrak{g}$, when Characteristic F is not 2.

Recall also that for a Lie algebra $\mathfrak{g}$, the Lie algebra homology of $\mathfrak{g}$ with coefficients in $\mathbb{R}$, written $H_*^{Lie}(\mathfrak{g};\mathbb{R})$ is the homology of the Chevalley-Eilenberg complex $\wedge^*(\mathfrak{g})$, namely

$$\mathbb{R} \overset{\partial}{\leftarrow} \mathfrak{g} \overset{\partial}{\leftarrow} \mathfrak{g}^{\wedge^2} \overset{\partial}{\leftarrow} \cdots \overset{\partial}{\leftarrow} \mathfrak{g}^{\wedge^{n-1}} \overset{\partial}{\leftarrow} \mathfrak{g}^{\wedge^n} \overset{\partial}{\leftarrow} \cdots$$

where $\mathfrak{g}^{\wedge^n}$ is the nth exterior power of g over k, and where

$$\partial(g_1 \wedge \cdots \wedge g_n) = \sum_{1 \leq i < j \leq n} (-1)^{i+j-1}[g_i, g_j] \wedge g_1 \wedge \cdots \hat{g}_i \cdots \hat{g}_j \cdots \wedge g_n$$

where $\hat{g}_i$ means that the variable g_i is deleted.

$$H_*^{Lie}(\mathfrak{g};\mathbb{R}) = \frac{ker(\partial_*)}{Im(\partial_{*+1})} \tag{1}$$

The following theeorem was given by Novica Blazic and Srdjan Vukmirovic [1], and classifies 4-dimensional Lie algebras with a Para-Hypercomplex up to isomorphism .

Theorem 2.1 *(Bianchi): Let $\mathfrak{g}$ be a real 4-dimensional Lie algebra. Then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras:*

1. $r_{4,\lambda}$: $[e_4, e_1] = e_1, [e_4, e_2] = \lambda e_2, [e_4, e_3] = e_2 + \lambda e_3$
2. $\mathfrak{h}_4$: $[e_4, e_3] = e_3, [e_1, e_2] = e_3, [e_4, e_2] = \frac{e_2}{2}, [e_4, e_1] = e_2 + \frac{e_1}{2}$
3. $\mathbb{R} \oplus \mathfrak{h}_3$: $[e_1, e_2] = e_3$
4. $\mathbb{R}^2 \oplus aff(\mathbb{R})$: $[e_1, e_2] = e_1$
5. $\mathbb{R} \oplus r_{3,1}$: $[e_1, e_2] = e_2, [e_1, e_4] = e_4$
6. $\mathbb{R} \oplus sl_2(\mathbb{R})$: $[e_1, e_2] = e_4, [e_2, e_4] = -e_1, [e_4, e_1] = e_2$
7. $aff(\mathbb{R}) \oplus aff(\mathbb{R})$: $[e_1, e_3] = e_1, [e_2, e_4] = e_2$
8. $t_{4,1,\lambda}$: $[e_4, e_1] = e_1, [e_4, e_2] = e_2, [e_4, e_3] = \lambda e_3$
9. $\delta_{4,\lambda}$: $[e_4, e_3] = e_3, [e_1, e_2] = e_3, [e_4, e_2] = (1-\lambda)e_2, [e_4, e_1] = \lambda e_1$
10. $aff(\mathbb{C})$: $[e_4, e_3] = e_3, [e_1, e_2] = e_3, [e_1, e_3] = -e_2, [e_4, e_2] = e_2$
11. δ_4 : $[e_1, e_2] = e_1, [e_2, e_4] = e_4, [e_1, e_4] = e_3$

3 Homology of Four-Dimensional Lie Algebras With a Para-Hypercomplex Structure

The following theorems provide the homology for each Lie algebras in the theorem 2.1.

Theorem 3.1 *Let* $\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$ *be a Lie algebra isomorphic to* $r_{4,\lambda}$, *given by the brackets* $[e_4, e_1] = e_1, [e_4, e_2] = \lambda e_2, [e_4, e_3] = e_2 + \lambda e_3$. *Then,*

$$H_k^{lie}(\mathfrak{g}) = \begin{cases} \mathbb{R} & k = 0 \\ < e_4 > & k = 1 \\ 0 & k \geq 2 \end{cases}$$

Proof.

The Chevalley-Eilenberg complex is reduced to
$0 \overset{\partial_0}{\longleftarrow} \mathbb{R} \overset{\partial_1}{\longleftarrow} \mathfrak{g} \overset{\partial_2}{\longleftarrow} \mathfrak{g}^{\wedge^2} \overset{\partial_3}{\longleftarrow} \mathfrak{g}^{\wedge^3} \leftarrow \mathfrak{g} \overset{\partial_4}{\longleftarrow} \mathfrak{g}^{\wedge^4} \leftarrow 0$

$\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$
$\mathfrak{g}^{\wedge 2} = span\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$
$\mathfrak{g}^{\wedge 3} = span\{e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4\}$
$\mathfrak{g}^{\wedge 4} = span\{e_1 \wedge e_2 \wedge e_3 \wedge e_4\}$

First we must evaluate ∂_k.

$\partial_0 = 0, \quad \partial_1 = 0, \quad \partial_2(e_1 \wedge e_2) = [e_1, e_2] = 0, \quad \partial_2(e_1 \wedge e_3) = [e_1, e_3] = 0,$
$\partial_2(e_1 \wedge e_4) = [e_1, e_4] = -[e_4, e_1] = -e_1, \quad \partial_2(e_2 \wedge e_3) = [e_2, e_3] = 0,$
$\partial_2(e_2 \wedge e_4) = [e_2, e_4] = -[e_4, e_2] = -\lambda e_2,$
$\partial_2(e_3 \wedge e_4) = [e_3, e_4] = -[e_4, e_3] = -e_2 - \lambda e_3,$
$\partial_3(e_1 \wedge e_2 \wedge e_3) = [e_1, e_2] \wedge e_3 - [e_1, e_3] \wedge e_2 + [e_2, e_3] \wedge e_1 = 0,$
$\partial_3(e_1 \wedge e_2 \wedge e_4) = [e_1, e_2] \wedge e_4 - [e_1, e_4] \wedge e_2 + [e_2, e_4] \wedge e_1 = [e_4, e_1] \wedge e_2 - [e_4, e_2] \wedge e_1 = e_1 \wedge e_2 + \lambda e_1 \wedge e_2 = (\lambda + 1) e_1 \wedge e_2,$
$\partial_3(e_1 \wedge e_3 \wedge e_4) = [e_1, e_3] \wedge e_4 - [e_1, e_4] \wedge e_3 + [e_3, e_4] \wedge e_1 = [e_4, e_1] \wedge e_3 - [e_4, e_3] \wedge e_1 = e_1 \wedge e_3 - (e_2 + \lambda e_3) \wedge e_1 = e_1 \wedge e_3 + e_1 \wedge e_2 + \lambda e_1 \wedge e_3 = e_1 \wedge e_2 + (\lambda + 1) e_1 \wedge e_3,$
$\partial_3(e_2 \wedge e_3 \wedge e_4) = [e_2, e_3] \wedge e_4 - [e_2, e_4] \wedge e_3 + [e_3, e_4] \wedge e_2 = \lambda e_2 \wedge e_3 - (e_2 + \lambda e_3) \wedge e_2 = 2\lambda e_2 \wedge e_3,$
$\partial_4(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = [e_1, e_2] \wedge e_3 \wedge e_4 - [e_1, e_3] \wedge e_2 \wedge e_4 + [e_1, e_4] \wedge e_2 \wedge e_3 + [e_2, e_3] \wedge e_1 \wedge e_4 - [e_2, e_4] \wedge e_1 \wedge e_3 + [e_3, e_4] \wedge e_1 \wedge e_2 = -[e_4, e_1] \wedge e_2 \wedge e_3 + [e_4, e_2] \wedge e_1 \wedge e_3 - [e_4, e_3] \wedge e_1 \wedge e_2 = -e_1 \wedge e_2 \wedge e_3 - \lambda e_1 \wedge e_2 \wedge e_3 - (e_2 + \lambda e_3) \wedge e_1 \wedge e_2 = -(2\lambda + 1) e_1 \wedge e_2 \wedge e_3.$

Now we may determine $\mathfrak{Im}\partial_k$.
$\mathfrak{Im}\partial_1 = 0,$
$\mathfrak{Im}\partial_2 =< -e_1, -\lambda e_2, -e_2 - \lambda e_3 >=< e_1, e_2, e_2 + \lambda e_3 >=< e_1, e_2, e_3 >,$

$\mathfrak{Im}\partial_3 =< (\lambda+1)e_1\wedge e_2, e_1\wedge e_2+(\lambda+1)e_1\wedge e_3, 2\lambda e_2\wedge e_3 >=< e_1\wedge e_2, e_1\wedge e_3, e_2\wedge e_3 >$,
$\mathfrak{Im}\partial_4 =< -(2\lambda+1)e_1\wedge e_2\wedge e_3 >=< e_1\wedge e_2\wedge e_3 >$.

Similarly, we now may determine $\ker\partial_k$.

$$\begin{aligned}
\ker\partial_0 &= \mathbb{R},\\
\ker\partial_1 &= g,\\
\ker\partial_2 &=< e_1\wedge e_2, e_1\wedge e_3, e_2\wedge e_3 >,\\
\ker\partial_3 &=< e_1\wedge e_2\wedge e_3 >,\\
\ker\partial_4 &= 0.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
H_0^{lie}(\mathfrak{g}) &= \frac{\ker\partial_0}{\mathfrak{Im}\partial_1}\\
&= \frac{\mathbb{R}}{0}\\
&= \mathbb{R},\\
H_1^{lie}(\mathfrak{g}) &= \frac{\ker\partial_1}{\mathfrak{Im}\partial_2}\\
&= \frac{g}{< e_1, e_2, e_3 >}\\
&=< e_4 >,\\
H_2^{lie}(\mathfrak{g}) &= \frac{\ker\partial_2}{\mathfrak{Im}\partial_3}\\
&= \frac{< e_1\wedge e_2, e_1\wedge e_3, e_2\wedge e_3 >}{< e_1\wedge e_2, e_1\wedge e_3, e_2\wedge e_3 >}\\
&= 0,\\
H_3^{lie}(\mathfrak{g}) &= \frac{\ker\partial_3}{\mathfrak{Im}\partial_4}\\
&= \frac{< e_1\wedge e_2\wedge e_3 >}{< e_1\wedge e_2\wedge e_3 >}\\
&= 0,\\
H_4^{lie}(\mathfrak{g}) &= \frac{\ker\partial_4}{\mathfrak{Im}\partial_5}\\
&= \frac{0}{0}\\
&= 0.
\end{aligned}$$

Therefore,

$$H_k^{lie}(\mathfrak{g}) = \begin{cases} \mathbb{R} & k=0\\ < e_4 > & k=1\\ 0 & k\geq 2 \end{cases}$$

.

□

Theorem 3.2 *Let* $\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$ *be a Lie algebra isomorphic to* h_4*, given by the brackets* $[e_4, e_3] = e_3, [e_1, e_2] = e_3, [e_4, e_2] = \frac{e_2}{2}, [e_4, e_1] = \frac{e_1}{2}$. *Then,*

$$H_k^{lie}(\mathfrak{g}) = \begin{cases} \mathbb{R} & k = 0 \\ < e_4 > & k = 1 \\ 0 & k \geq 2 \end{cases}$$

Proof.

The Chevalley-Eilenberg complex is reduced to
$0 \xleftarrow{\partial_0} \mathbb{R} \xleftarrow{\partial_1} \mathfrak{g} \xleftarrow{\partial_2} \mathfrak{g}^{\wedge^2} \xleftarrow{\partial_3} \mathfrak{g}^{\wedge^3} \leftarrow \mathfrak{g} \xleftarrow{\partial_4} \mathfrak{g}^{\wedge^4} \leftarrow 0$

$\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$
$\mathfrak{g}^{\wedge 2} = span\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$
$\mathfrak{g}^{\wedge 3} = span\{e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4\}$
$\mathfrak{g}^{\wedge 4} = span\{e_1 \wedge e_2 \wedge e_3 \wedge e_4\}$

$$\begin{aligned} \partial_0 &= 0, \\ \partial_1 &= 0, \\ \partial_2(e_1 \wedge e_2) &= [e_1, e_2] \\ &= e_3, \\ \partial_2(e_1 \wedge e_3) &= [e_1, e_3] \\ &= 0, \\ \partial_2(e_1 \wedge e_4) &= [e_1, e_4] \\ &= -[e_4, e_1] \\ &= -e_2 - \frac{e_1}{2}, \\ \partial_2(e_2 \wedge e_3) &= [e_2, e_3] \\ &= 0, \\ \partial_2(e_2 \wedge e_4) &= [e_2, e_4] \\ &= -[e_4, e_2] \\ &= -\frac{e_2}{2}, \\ \partial_2(e_3 \wedge e_4) &= [e_3, e_4] \\ &= -[e_4, e_3] \\ &= -e_3, \\ \partial_3(e_1 \wedge e_2 \wedge e_3) &= [e_1, e_2] \wedge e_3 - [e_1, e_3] \wedge e_2 + [e_2, e_3] \wedge e_1 \\ &= 0, \end{aligned}$$

$$
\begin{aligned}
\partial_3(e_1 \wedge e_2 \wedge e_4) &= [e_1, e_2] \wedge e_4 - [e_1, e_4] \wedge e_2 + [e_2, e_4] \wedge e_1 \\
&= [e_1, e_2] \wedge e_4 + [e_4, e_1] \wedge e_2 - [e_4, e_2] \wedge e_1 \\
&= e_3 \wedge e_4 + (e_2 + \frac{e_1}{2}) \wedge e_2 - \frac{e_2}{2} \wedge e_1 \\
&= e_1 \wedge \frac{e_2}{2} + \frac{e_1}{2} \wedge e_2 + e_3 \wedge e_4 \\
&= e_1 \wedge e_2 - e_4 \wedge e_3, \\
\partial_3(e_1 \wedge e_3 \wedge e_4) &= [e_1, e_3] \wedge e_4 - [e_1, e_4] \wedge e_3 + [e_3, e_4] \wedge e_1 \\
&= [e_4, e_1] \wedge e_3 - [e_4, e_3] \wedge e_1 \\
&= (e_2 + \frac{e_1}{2}) \wedge e_3 - e_3 \wedge e_1 \\
&= e_2 \wedge e_3 + \frac{e_1}{2} \wedge e_3 + e_1 \wedge e_3 \\
&= \frac{3}{2} e_1 \wedge e_3 + e_2 \wedge e_3, \\
\partial_3(e_2 \wedge e_3 \wedge e_4) &= [e_2, e_3] \wedge e_4 - [e_2, e_4] \wedge e_3 + [e_3, e_4] \wedge e_2 \\
&= [e_4, e_2] \wedge e_3 - [e_4, e_3] \wedge e_2 \\
&= \frac{e_2}{2} \wedge e_3 + e_2 \wedge e_3 \\
&= \frac{3}{2} e_2 \wedge e_3, \\
\partial_4(e_1 \wedge e_2 \wedge e_3 \wedge e_4) &= [e_1, e_2] \wedge e_3 \wedge e_4 - [e_1, e_3] \wedge e_2 \wedge e_4 + [e_1, e_4] \wedge e_2 \wedge e_3 \\
&\quad + [e_2, e_3] \wedge e_1 \wedge e_4 - [e_2, e_4] \wedge e_1 \wedge e_3 + [e_3, e_4] \wedge e_1 \wedge e_2 \\
&= -[e_4, e_1] \wedge e_2 \wedge e_3 + [e_4, e_2] \wedge e_1 \wedge e_3 - [e_4, e_3] \wedge e_1 \wedge e_2 \\
&= -(e_2 + \frac{e_1}{2}) \wedge e_2 \wedge e_3 + \frac{1}{2} e_2 \wedge e_1 \wedge e_3 - e_3 \wedge e_1 \wedge e_2 \\
&= -e_1 \wedge e_2 \wedge e_3.
\end{aligned}
$$

Now we may determine $\mathfrak{Im}\partial_k$.

$$
\begin{aligned}
\mathfrak{Im}\partial_1 &= 0, \\
\mathfrak{Im}\partial_2 &=< e_3, -e_2 - \frac{e_1}{2}, -\frac{e_2}{2} > \\
&=< e_1, e_2, e_3 >, \\
\mathfrak{Im}\partial_3 &=< e_1 \wedge e_2 + e_4 \wedge e_3, \frac{3}{2} e_1 \wedge e_3 + e_2 \wedge e_3, \frac{3}{2} e_2 \wedge e_3 > \\
&=< e_1 \wedge e_2 - e_4 \wedge e_3, e_1 \wedge e_3, e_2 \wedge e_3 > \\
\mathfrak{Im}\partial_4 &=< -e_1 \wedge e_2 \wedge e_3 > \\
&=< e_1 \wedge e_2 \wedge e_3 > .
\end{aligned}
$$

Similarly, we now may determine $\ker \partial_k$.

$$\ker \partial_0 = \mathbb{R},$$

$$\ker \partial_1 = g,$$
$$\ker \partial_2 =< e_1 \wedge e_3, e_2 \wedge e_3, e_1 \wedge e_2 - e_4 \wedge e_3 >,$$
$$\ker \partial_3 =< e_1 \wedge e_2 \wedge e_3 >,$$
$$\ker \partial_4 = 0.$$

Thus, we have

$$\begin{aligned}
H_0^{lie}(\mathfrak{g}) &= \frac{\ker \partial_0}{\mathfrak{Im}\partial_1} \\
&= \frac{\mathbb{R}}{0} \\
&= \mathbb{R}, \\
H_1^{lie}(\mathfrak{g}) &= \frac{\ker \partial_1}{\mathfrak{Im}\partial_2} \\
&= \frac{g}{< e_1, e_2, e_3 >} \\
&=< e_4 >, \\
H_2^{lie}(\mathfrak{g}) &= \frac{\ker \partial_2}{\mathfrak{Im}\partial_3} \\
&= \frac{< e_1 \wedge e_3, e_2 \wedge e_3, e_1 \wedge e_2 - e_4 \wedge e_3 >}{< e_1 \wedge e_2 - e_4 \wedge e_3, e_1 \wedge e_3, e_2 \wedge e_3 >} \\
&= 0 \\
H_3^{lie}(\mathfrak{g}) &= \frac{\ker \partial_3}{\mathfrak{Im}\partial_4} \\
&= \frac{< e_1 \wedge e_2 \wedge e_3 >}{< e_1 \wedge e_2 \wedge e_3 >} \\
&= 0, \\
H_4^{lie}(\mathfrak{g}) &= \frac{\ker \partial_4}{\mathfrak{Im}\partial_5} \\
&= \frac{0}{0} \\
&= 0.
\end{aligned}$$

Therefore,

$$H_k^{lie}(\mathfrak{g}) = \begin{cases} \mathbb{R} & k = 0 \\ < e_4 > & k = 1 \\ 0 & k \geq 2 \end{cases}$$

.

Theorem 3.3 *Let* $\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$ *be a Lie algebra isomorphic to* $\mathbb{R} \bigoplus h_3$, *given by the brackets* $[e_1, e_2] = e_3$. *Then,*

$$H_k^{lie}(\mathfrak{g}) = \begin{cases} \mathbb{R} & k = 0 \\ < e_1, e_2, e_4 > & k = 1 \\ < e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4 > & k = 2, \\ < e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4 > & k = 3, \\ < e_1 \wedge e_2 \wedge e_3 \wedge e_4 > & k = 4, \\ 0 & k \geq 5. \end{cases}$$

Proof.

The Chevalley-Eilenberg complex is reduced to

$0 \overset{\partial_0}{\longleftarrow} \mathbb{R} \overset{\partial_1}{\longleftarrow} \mathfrak{g} \overset{\partial_2}{\longleftarrow} \mathfrak{g}^{\wedge^2} \overset{\partial_3}{\longleftarrow} \mathfrak{g}^{\wedge^3} \leftarrow \mathfrak{g} \overset{\partial_4}{\longleftarrow} \mathfrak{g}^{\wedge^4} \leftarrow 0$

$\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$

$\mathfrak{g}^{\wedge 2} = span\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$

$\mathfrak{g}^{\wedge 3} = span\{e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4\}$

$\mathfrak{g}^{\wedge 4} = span\{e_1 \wedge e_2 \wedge e_3 \wedge e_4\}$

First we must evaluate ∂_k.

$$\begin{aligned} \partial_0 &= 0, \\ \partial_1 &= 0, \\ \partial_2(e_1 \wedge e_2) &= [e_1, e_2] \\ &= e_3, \\ \partial_2(e_1 \wedge e_3) &= [e_1, e_3] \\ &= 0, \\ \partial_2(e_1 \wedge e_4) &= [e_1, e_4] \\ &= 0, \\ \partial_2(e_2 \wedge e_3) &= [e_2, e_3] \\ &= 0, \\ \partial_2(e_2 \wedge e_4) &= [e_2, e_4] \\ &= 0, \\ \partial_2(e_3 \wedge e_4) &= [e_3, e_4] \\ &= 0, \\ \partial_3(e_1 \wedge e_2 \wedge e_3) &= [e_1, e_2] \wedge e_3 - [e_1, e_3] \wedge e_2 + [e_2, e_3] \wedge e_1 \\ &= 0, \\ \partial_3(e_1 \wedge e_2 \wedge e_4) &= [e_1, e_2] \wedge e_4 - [e_1, e_4] \wedge e_2 + [e_2, e_4] \wedge e_1 \\ &= e_3 \wedge e_4, \\ \partial_3(e_1 \wedge e_3 \wedge e_4) &= [e_1, e_3] \wedge e_4 - [e_1, e_4] \wedge e_3 + [e_3, e_4] \wedge e_1 \\ &= 0, \end{aligned}$$

$$\begin{aligned}\partial_3(e_2 \wedge e_3 \wedge e_4) &= [e_2, e_3] \wedge e_4 - [e_2, e_4] \wedge e_3 + [e_3, e_4] \wedge e_2 \\ &= 0, \\ \partial_4(e_1 \wedge e_2 \wedge e_3 \wedge e_4) &= [e_1, e_2] \wedge e_3 \wedge e_4 - [e_1, e_3] \wedge e_2 \wedge e_4 + [e_1, e_4] \wedge e_2 \wedge e_3 \\ &+ [e_2, e_3] \wedge e_1 \wedge e_4 - [e_2, e_4] \wedge e_1 \wedge e_3 + [e_3, e_4] \wedge e_1 \wedge e_2 \\ &= 0.\end{aligned}$$

Now we may determine $\mathfrak{Im}\partial_k$.

$$\begin{aligned}\mathfrak{Im}\partial_1 &= 0, \\ \mathfrak{Im}\partial_2 &=< e_3 >, \\ \mathfrak{Im}\partial_3 &=< e_3 \wedge e_4 >, \\ \mathfrak{Im}\partial_4 &= 0.\end{aligned}$$

Similarly, we now may determine $\ker \partial_k$.

$$\begin{aligned}&\ker \partial_0 = \mathbb{R}, \\ &\ker \partial_1 = g, \\ &\ker \partial_2 =< e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4 >, \\ &\ker \partial_3 =< e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4 >, \\ &\ker \partial_4 =< e_1 \wedge e_2 \wedge e_3 \wedge e_4 > .\end{aligned}$$

Thus, we have

$$\begin{aligned}H_0^{lie}(\mathfrak{g}) &= \frac{\ker \partial_0}{\mathfrak{Im}\partial_1} \\ &= \frac{\mathbb{R}}{0} \\ &= \mathbb{R}, \\ H_1^{lie}(\mathfrak{g}) &= \frac{\ker \partial_1}{\mathfrak{Im}\partial_2} \\ &= \frac{g}{< e_3 >} \\ &=< e_1, e_2, e_4 > \\ H_2^{lie}(\mathfrak{g}) &= \frac{\ker \partial_2}{\mathfrak{Im}\partial_3} \\ &= \frac{< e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4 >}{< e_3 \wedge e_4 >} \\ &=< e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4 >, \\ H_3^{lie}(\mathfrak{g}) &= \frac{\ker \partial_3}{\mathfrak{Im}\partial_4}\end{aligned}$$

$$= \frac{< e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4 >}{0}$$
$$=< e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4 >,$$
$$H_4^{lie}(\mathfrak{g}) = \frac{\ker \partial_4}{\mathfrak{Im}\partial_5}$$
$$= \frac{< e_1 \wedge e_2 \wedge e_3 \wedge e_4 >}{0}$$
$$=< e_1 \wedge e_2 \wedge e_3 \wedge e_4 > .$$

Therefore,

$$H_k^{lie}(\mathfrak{g}) = \begin{cases} \mathbb{R} & k = 0 \\ < e_1, e_2, e_4 > & k = 1 \\ < e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4 > & k = 2, \\ < e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4 > & k = 3, \\ < e_1 \wedge e_2 \wedge e_3 \wedge e_4 > & k = 4, \\ 0 & k \geq 5. \end{cases}$$

.

Theorem 3.4 *Let* $\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$ *be a Lie algebra isomorphic to* $\mathbb{R}^2 \bigoplus aff(\mathbb{R},$ *given by the brackets* $[e_1, e_2] = e_1$. *Then,*

$$H_k^{lie}(\mathfrak{g}) = \begin{cases} \mathbb{R} & k = 0 \\ < e_2, e_3, e_4 > & k = 1 \\ < e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4 > & k = 2, \\ < e_2 \wedge e_3 \wedge e_4 > & k = 3, \\ 0 & k \geq 4. \end{cases}$$

Proof.

The Chevalley-Eilenberg complex is reduced to
$0 \overset{\partial_0}{\longleftarrow} \mathbb{R} \overset{\partial_1}{\longleftarrow} \mathfrak{g} \overset{\partial_2}{\longleftarrow} \mathfrak{g}^{\wedge^2} \overset{\partial_3}{\longleftarrow} \mathfrak{g}^{\wedge^3} \leftarrow \mathfrak{g} \overset{\partial_4}{\longleftarrow} \mathfrak{g}^{\wedge^4} \leftarrow 0$

$\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$
$\mathfrak{g}^{\wedge 2} = span\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$
$\mathfrak{g}^{\wedge 3} = span\{e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4\}$
$\mathfrak{g}^{\wedge 4} = span\{e_1 \wedge e_2 \wedge e_3 \wedge e_4\}$
First we must evaluate ∂_k.

$$\partial_0 = 0,$$
$$\partial_1 = 0,$$
$$\partial_2(e_1 \wedge e_2) = [e_1, e_2]$$

$$
\begin{aligned}
&= e_1, \\
\partial_2(e_1 \wedge e_3) &= [e_1, e_3] \\
&= 0, \\
\partial_2(e_1 \wedge e_4) &= [e_1, e_4] \\
&= 0, \\
\partial_2(e_2 \wedge e_3) &= [e_2, e_3] \\
&= 0, \\
\partial_2(e_2 \wedge e_4) &= [e_2, e_4] \\
&= 0, \\
\partial_2(e_3 \wedge e_4) &= [e_3, e_4] \\
&= 0, \\
\partial_3(e_1 \wedge e_2 \wedge e_3) &= [e_1, e_2] \wedge e_3 - [e_1, e_3] \wedge e_2 + [e_2, e_3] \wedge e_1 \\
&= e_1 \wedge e_3, \\
\partial_3(e_1 \wedge e_2 \wedge e_4) &= [e_1, e_2] \wedge e_4 - [e_1, e_4] \wedge e_2 + [e_2, e_4] \wedge e_1 \\
&= e_1 \wedge e_4, \\
\partial_3(e_1 \wedge e_3 \wedge e_4) &= [e_1, e_3] \wedge e_4 - [e_1, e_4] \wedge e_3 + [e_3, e_4] \wedge e_1 \\
&= 0, \\
\partial_3(e_2 \wedge e_3 \wedge e_4) &= [e_2, e_3] \wedge e_4 - [e_2, e_4] \wedge e_3 + [e_3, e_4] \wedge e_2 \\
&= 0, \\
\partial_4(e_1 \wedge e_2 \wedge e_3 \wedge e_4) &= [e_1, e_2] \wedge e_3 \wedge e_4 - [e_1, e_3] \wedge e_2 \wedge e_4 + [e_1, e_4] \wedge e_2 \wedge e_3 \\
&\quad + [e_2, e_3] \wedge e_1 \wedge e_4 - [e_2, e_4] \wedge e_1 \wedge e_3 + [e_3, e_4] \wedge e_1 \wedge e_2 \\
&= e_1 \wedge e_3 \wedge e_4.
\end{aligned}
$$

Now we may determine $\mathfrak{Im}\partial_k$.

$$
\begin{aligned}
&\mathfrak{Im}\partial_1 = 0, \\
&\mathfrak{Im}\partial_2 =< e_1 >, \\
&\mathfrak{Im}\partial_3 =< e_1 \wedge e_3, e_1 \wedge e_4 >, \\
&\mathfrak{Im}\partial_4 =< e_1 \wedge e_3 \wedge e_4 > .
\end{aligned}
$$

Similarly, we now may determine $\ker \partial_k$.

$$
\begin{aligned}
&\ker \partial_0 = \mathbb{R}, \\
&\ker \partial_1 = g, \\
&\ker \partial_2 =< e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4 >, \\
&\ker \partial_3 =< e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4 >,
\end{aligned}
$$

$\ker \partial_4 = 0.$

Thus, we have

$$\begin{aligned}
H_0^{lie}(\mathfrak{g}) &= \frac{\ker \partial_0}{\mathfrak{Im}\partial_1} \\
&= \frac{\mathbb{R}}{0} \\
&= \mathbb{R}, \\
H_1^{lie}(\mathfrak{g}) &= \frac{\ker \partial_1}{\mathfrak{Im}\partial_2} \\
&= \frac{g}{< e_1 >} \\
&=< e_2, e_3, e_4 > \\
H_2^{lie}(\mathfrak{g}) &= \frac{\ker \partial_2}{\mathfrak{Im}\partial_3} \\
&= \frac{< e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4 >}{< e_1 \wedge e_3, e_1 \wedge e_4 >} \\
&=< e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4 >, \\
H_3^{lie}(\mathfrak{g}) &= \frac{\ker \partial_3}{\mathfrak{Im}\partial_4} \\
&= \frac{< e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4 >}{< e_1 \wedge e_3 \wedge e_4 >} \\
&=< e_2 \wedge e_3 \wedge e_4 >, \\
H_4^{lie}(\mathfrak{g}) &= \frac{\ker \partial_4}{\mathfrak{Im}\partial_5} \\
&= \frac{0}{0} \\
&= 0
\end{aligned}$$

Therefore,

$$H_k^{lie}(\mathfrak{g}) = \begin{cases} \mathbb{R} & k = 0 \\ < e_2, e_3, e_4 > & k = 1 \\ < e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4 > & k = 2, \\ < e_2 \wedge e_3 \wedge e_4 > & k = 3, \\ 0 & k \geq 4 \end{cases}$$

.

Theorem 3.5 *Let $\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$ be a Lie algebra isomorphic to $\mathbb{R} \bigoplus r_{3,1}$,* *given by the brackets* $[e_1, e_2] = e_2, [e_1, e_4] = e_4$. *Then,*

$$H_k^{lie}(\mathfrak{g}) = \begin{cases} \mathbb{R} & k = 0 \\ < e_1, e_3 > & k = 1 \\ < e_1 \wedge e_3 > & k = 2, \\ < e_2 \wedge e_3 \wedge e_4 > & k = 3, \\ < e_1 \wedge e_2 \wedge e_3 \wedge e_4 > & k = 4, \\ 0 & k \geq 5. \end{cases}$$

Proof.

The Chevalley-Eilenberg complex is reduced to

$0 \xleftarrow{\partial_0} \mathbb{R} \xleftarrow{\partial_1} \mathfrak{g} \xleftarrow{\partial_2} \mathfrak{g}^{\wedge^2} \xleftarrow{\partial_3} \mathfrak{g}^{\wedge^3} \leftarrow \mathfrak{g} \xleftarrow{\partial_4} \mathfrak{g}^{\wedge^4} \leftarrow 0$

$\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$
$\mathfrak{g}^{\wedge 2} = span\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$
$\mathfrak{g}^{\wedge 3} = span\{e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4\}$
$\mathfrak{g}^{\wedge 4} = span\{e_1 \wedge e_2 \wedge e_3 \wedge e_4\}$

$$\begin{aligned} \partial_0 &= 0, \\ \partial_1 &= 0, \\ \partial_2(e_1 \wedge e_2) &= [e_1, e_2] \\ &= e_2, \\ \partial_2(e_1 \wedge e_3) &= [e_1, e_3] \\ &= 0, \\ \partial_2(e_1 \wedge e_4) &= [e_1, e_4] \\ &= e_4, \\ \partial_2(e_2 \wedge e_3) &= [e_2, e_3] \\ &= 0, \\ \partial_2(e_2 \wedge e_4) &= [e_2, e_4] \\ &= 0, \\ \partial_2(e_3 \wedge e_4) &= [e_3, e_4] \\ &= 0, \\ \partial_3(e_1 \wedge e_2 \wedge e_3) &= [e_1, e_2] \wedge e_3 - [e_1, e_3] \wedge e_2 + [e_2, e_3] \wedge e_1 \\ &= e_2 \wedge e_3, \\ \partial_3(e_1 \wedge e_2 \wedge e_4) &= [e_1, e_2] \wedge e_4 - [e_1, e_4] \wedge e_2 + [e_2, e_4] \wedge e_1 \\ &= e_2 \wedge e_4 - e_4 \wedge e_2 \\ &= 2e_2 \wedge e_4, \\ \partial_3(e_1 \wedge e_3 \wedge e_4) &= [e_1, e_3] \wedge e_4 - [e_1, e_4] \wedge e_3 + [e_3, e_4] \wedge e_1 \end{aligned}$$

$$
\begin{aligned}
&= e_3 \wedge e_4, \\
\partial_3(e_2 \wedge e_3 \wedge e_4) &= [e_2, e_3] \wedge e_4 - [e_2, e_4] \wedge e_3 + [e_3, e_4] \wedge e_2 \\
&= 0, \\
\partial_4(e_1 \wedge e_2 \wedge e_3 \wedge e_4) &= [e_1, e_2] \wedge e_3 \wedge e_4 - [e_1, e_3] \wedge e_2 \wedge e_4 + [e_1, e_4] \wedge e_2 \wedge e_3 \\
&+ [e_2, e_3] \wedge e_1 \wedge e_4 - [e_2, e_4] \wedge e_1 \wedge e_3 + [e_3, e_4] \wedge e_1 \wedge e_2 \\
&= e_2 \wedge e_3 \wedge e_4 - e_4 \wedge e_2 \wedge e_3 \\
&= 0.
\end{aligned}
$$

Now we may determine $\mathfrak{Im}\partial_k$.

$$
\begin{aligned}
\mathfrak{Im}\partial_1 &= 0, \\
\mathfrak{Im}\partial_2 &=< e_2, e_4 > \\
\mathfrak{Im}\partial_3 &=< e_2 \wedge e_3, 2e_2 \wedge e_4, e_3 \wedge e_4 > \\
&=< e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4 > \\
\mathfrak{Im}\partial_4 &= 0.
\end{aligned}
$$

Similarly, we now may determine $\ker \partial_k$.

$$
\begin{aligned}
&\ker \partial_0 = \mathbb{R}, \\
&\ker \partial_1 = g, \\
&\ker \partial_2 =< e_1 \wedge e_3, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4 >, \\
&\ker \partial_3 =< e_2 \wedge e_3 \wedge e_4 >, \\
&\ker \partial_4 =< e_1 \wedge e_2 \wedge e_3 \wedge e_4 > .
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
H_0^{lie}(\mathfrak{g}) &= \frac{\ker \partial_0}{\mathfrak{Im}\partial_1} \\
&= \frac{\mathbb{R}}{0} \\
&= \mathbb{R}, \\
H_1^{lie}(\mathfrak{g}) &= \frac{\ker \partial_1}{\mathfrak{Im}\partial_2} \\
&= \frac{g}{< e_2, e_4 >} \\
&=< e_1, e_3 > \\
H_2^{lie}(\mathfrak{g}) &= \frac{\ker \partial_2}{\mathfrak{Im}\partial_3}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{< e_1 \wedge e_3, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4 >}{< e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4 >} \\
&=< e_1 \wedge e_3 >, \\
H_3^{lie}(\mathfrak{g}) &= \frac{\ker \partial_3}{\mathfrak{Im} \partial_4} \\
&= \frac{< e_2 \wedge e_3 \wedge e_4 >}{0} \\
&=< e_2 \wedge e_3 \wedge e_4 >, \\
H_4^{lie}(\mathfrak{g}) &= \frac{\ker \partial_4}{\mathfrak{Im} \partial_5} \\
&= \frac{< e_1 \wedge e_2 \wedge e_3 \wedge e_4 >}{0} \\
&=< e_1 \wedge e_2 \wedge e_3 \wedge e_4 > .
\end{aligned}
$$

Therefore,

$$
H_k^{lie}(\mathfrak{g}) = \begin{cases} \mathbb{R} & k = 0 \\ < e_1, e_3 > & k = 1 \\ < e_1 \wedge e_3 > & k = 2, \\ < e_2 \wedge e_3 \wedge e_4 > & k = 3, \\ < e_1 \wedge e_2 \wedge e_3 \wedge e_4 > & k = 4, \\ 0 & k \geq 5. \end{cases}
$$

.

Theorem 3.6 *Let* $\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$ *be a Lie algebra isomorphic to* $\mathbb{R} \bigoplus sl_2(\mathbb{R})$, *given by the brackets* $[e_1, e_2] = e_4, [e_2, e_4] = -e_1, [e_4, e_1] = e_2$. *Then,*

$$
H_k^{lie}(\mathfrak{g}) = \begin{cases} \mathbb{R} & k = 0 \\ < e_3 > & k = 1 \\ 0 & k = 2, \\ < e_1 \wedge e_2 \wedge e_4 > & k = 3, \\ < e_1 \wedge e_2 \wedge e_3 \wedge e_4 > & k = 4, \\ 0 & k \geq 5. \end{cases}
$$

Proof.

The Chevalley-Eilenberg complex is reduced to
$0 \xleftarrow{\partial_0} \mathbb{R} \xleftarrow{\partial_1} \mathfrak{g} \xleftarrow{\partial_2} \mathfrak{g}^{\wedge^2} \xleftarrow{\partial_3} \mathfrak{g}^{\wedge^3} \leftarrow \mathfrak{g} \xleftarrow{\partial_4} \mathfrak{g}^{\wedge^4} \leftarrow 0$

$\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$
$\mathfrak{g}^{\wedge 2} = span\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$
$\mathfrak{g}^{\wedge 3} = span\{e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4\}$
$\mathfrak{g}^{\wedge 4} = span\{e_1 \wedge e_2 \wedge e_3 \wedge e_4\}$

First we must evaluate ∂_k.

$$
\begin{aligned}
\partial_0 &= 0,\\
\partial_1 &= 0,\\
\partial_2(e_1 \wedge e_2) &= [e_1, e_2]\\
&= e_4,\\
\partial_2(e_1 \wedge e_3) &= [e_1, e_3]\\
&= 0,\\
\partial_2(e_1 \wedge e_4) &= [e_1, e_4]\\
&= -[e_4, e_1],\\
&= -e_2,\\
\partial_2(e_2 \wedge e_3) &= [e_2, e_3]\\
&= 0,\\
\partial_2(e_2 \wedge e_4) &= [e_2, e_4]\\
&= -e_1,\\
\partial_2(e_3 \wedge e_4) &= [e_3, e_4]\\
&= 0,\\
\partial_3(e_1 \wedge e_2 \wedge e_3) &= [e_1, e_2] \wedge e_3 - [e_1, e_3] \wedge e_2 + [e_2, e_3] \wedge e_1\\
&= e_4 \wedge e_3,\\
\partial_3(e_1 \wedge e_2 \wedge e_4) &= [e_1, e_2] \wedge e_4 - [e_1, e_4] \wedge e_2 + [e_2, e_4] \wedge e_1\\
&= [e_1, e_2] \wedge e_4 + [e_4, e_1] \wedge e_2 + [e_2, e_4] \wedge e_1\\
&= 0,\\
\partial_3(e_1 \wedge e_3 \wedge e_4) &= [e_1, e_3] \wedge e_4 - [e_1, e_4] \wedge e_3 + [e_3, e_4] \wedge e_1\\
&= [e_1, e_3] \wedge e_4 + [e_4, e_1] \wedge e_3 + [e_3, e_4] \wedge e_1\\
&= e_2 \wedge e_3,\\
\partial_3(e_2 \wedge e_3 \wedge e_4) &= [e_2, e_3] \wedge e_4 - [e_2, e_4] \wedge e_3 + [e_3, e_4] \wedge e_2\\
&= e_1 \wedge e_3,\\
\partial_4(e_1 \wedge e_2 \wedge e_3 \wedge e_4) &= [e_1, e_2] \wedge e_3 \wedge e_4 - [e_1, e_3] \wedge e_2 \wedge e_4 + [e_1, e_4] \wedge e_2 \wedge e_3\\
&+ [e_2, e_3] \wedge e_1 \wedge e_4 - [e_2, e_4] \wedge e_1 \wedge e_3 + [e_3, e_4] \wedge e_1 \wedge e_2\\
&= [e_1, e_2] \wedge e_3 \wedge e_4 - [e_1, e_3] \wedge e_2 \wedge e_4 - [e_4, e_1] \wedge e_2 \wedge e_3\\
&+ [e_2, e_3] \wedge e_1 \wedge e_4 - [e_2, e_4] \wedge e_1 \wedge e_3 + [e_3, e_4] \wedge e_1 \wedge e_2\\
&= 0.
\end{aligned}
$$

Now we may determine $\mathfrak{Im}\partial_k$.

$$
\begin{aligned}
&\mathfrak{Im}\partial_1 = 0,\\
&\mathfrak{Im}\partial_2 =< e_4, e_2, -e_1 >
\end{aligned}
$$

$$
\begin{aligned}
&= < e_4, e_2, e_1 >, \\
\mathfrak{Im}\partial_3 &= < e_4 \wedge e_3, e_2 \wedge e_3, e_1 \wedge e_3 > \\
&= < e_1 \wedge e_3, e_2 \wedge e_3, e_3 \wedge e_4 > \\
\mathfrak{Im}\partial_4 &= 0.
\end{aligned}
$$

Similarly, we now may determine $\ker \partial_k$.

$$
\begin{aligned}
\ker \partial_0 &= \mathbb{R}, \\
\ker \partial_1 &= g, \\
\ker \partial_2 &= < e_1 \wedge e_3, e_2 \wedge e_3, e_3 \wedge e_4 >, \\
\ker \partial_3 &= < e_1 \wedge e_2 \wedge e_4 >, \\
\ker \partial_4 &= < e_1 \wedge e_2 \wedge e_3 \wedge e_4 > .
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
H_0^{lie}(\mathfrak{g}) &= \frac{\ker \partial_0}{\mathfrak{Im}\partial_1} \\
&= \frac{\mathbb{R}}{0} \\
&= \mathbb{R}, \\
H_1^{lie}(\mathfrak{g}) &= \frac{\ker \partial_1}{\mathfrak{Im}\partial_2} \\
&= \frac{g}{< e_4, e_2, e_1 >} \\
&= < e_3 > \\
H_2^{lie}(\mathfrak{g}) &= \frac{\ker \partial_2}{\mathfrak{Im}\partial_3} \\
&= \frac{< e_1 \wedge e_3, e_2 \wedge e_3, e_3 \wedge e_4 >}{< e_1 \wedge e_3, e_2 \wedge e_3, e_3 \wedge e_4 >} \\
&= 0, \\
H_3^{lie}(\mathfrak{g}) &= \frac{\ker \partial_3}{\mathfrak{Im}\partial_4} \\
&= \frac{< e_1 \wedge e_2 \wedge e_4 >}{0} \\
&= < e_1 \wedge e_2 \wedge e_4 >, \\
H_4^{lie}(\mathfrak{g}) &= \frac{\ker \partial_4}{\mathfrak{Im}\partial_5} \\
&= \frac{< e_1 \wedge e_2 \wedge e_3 \wedge e_4 >}{0} \\
&= < e_1 \wedge e_2 \wedge e_3 \wedge e_4 > .
\end{aligned}
$$

Therefore,

$$H_k^{lie}(\mathfrak{g}) = \begin{cases} \mathbb{R} & k=0 \\ < e_3 > & k=1 \\ 0 & k=2, \\ < e_1 \wedge e_2 \wedge e_4 > & k=3, \\ < e_1 \wedge e_2 \wedge e_3 \wedge e_4 > & k=4, \\ 0 & k \geq 5. \end{cases}$$

.

Theorem 3.7 *Let $\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$ be a Lie algebra isomorphic to $aff(\mathbb{R}) \bigoplus aff(\mathbb{R})$, given by the brackets $[e_1, e_3] = e_1, [e_2, e_4] = e_2$. Then,*

$$H_k^{lie}(\mathfrak{g}) = \begin{cases} \mathbb{R} & k=0 \\ < e_1, e_4 > & k=1 \\ < e_3 \wedge e_4 > & k=2, \\ 0 & k \geq 3 \end{cases}$$

Proof.

The Chevalley-Eilenberg complex is reduced to
$0 \overset{\partial_0}{\longleftarrow} \mathbb{R} \overset{\partial_1}{\longleftarrow} \mathfrak{g} \overset{\partial_2}{\longleftarrow} \mathfrak{g}^{\wedge^2} \overset{\partial_3}{\longleftarrow} \mathfrak{g}^{\wedge^3} \leftarrow \mathfrak{g} \overset{\partial_4}{\longleftarrow} \mathfrak{g}^{\wedge^4} \leftarrow 0$

$\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$
$\mathfrak{g}^{\wedge 2} = span\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$
$\mathfrak{g}^{\wedge 3} = span\{e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4\}$
$\mathfrak{g}^{\wedge 4} = span\{e_1 \wedge e_2 \wedge e_3 \wedge e_4\}$

$$\begin{aligned} \partial_0 &= 0, \\ \partial_1 &= 0, \\ \partial_2(e_1 \wedge e_2) &= [e_1, e_2] \\ &= 0, \\ \partial_2(e_1 \wedge e_3) &= [e_1, e_3] \\ &= e_1, \\ \partial_2(e_1 \wedge e_4) &= [e_1, e_4] \\ &= 0, \\ \partial_2(e_2 \wedge e_3) &= [e_2, e_3] \\ &= 0, \\ \partial_2(e_2 \wedge e_4) &= [e_2, e_4] \\ &= e_2, \\ \partial_2(e_3 \wedge e_4) &= [e_3, e_4] \end{aligned}$$

$$
\begin{aligned}
&= 0, \\
\partial_3(e_1 \wedge e_2 \wedge e_3) &= [e_1, e_2] \wedge e_3 - [e_1, e_3] \wedge e_2 + [e_2, e_3] \wedge e_1 \\
&= -e_1 \wedge e_2, \\
\partial_3(e_1 \wedge e_2 \wedge e_4) &= [e_1, e_2] \wedge e_4 - [e_1, e_4] \wedge e_2 + [e_2, e_4] \wedge e_1 \\
&= -e_1 \wedge e_2 \\
\partial_3(e_1 \wedge e_3 \wedge e_4) &= [e_1, e_3] \wedge e_4 - [e_1, e_4] \wedge e_3 + [e_3, e_4] \wedge e_1 \\
&= e_1 \wedge e_4, \\
\partial_3(e_2 \wedge e_3 \wedge e_4) &= [e_2, e_3] \wedge e_4 - [e_2, e_4] \wedge e_3 + [e_3, e_4] \wedge e_2 \\
&= -e_2 \wedge e_3, \\
\partial_4(e_1 \wedge e_2 \wedge e_3 \wedge e_4) &= [e_1, e_2] \wedge e_3 \wedge e_4 - [e_1, e_3] \wedge e_2 \wedge e_4 + [e_1, e_4] \wedge e_2 \wedge e_3 \\
&+ [e_2, e_3] \wedge e_1 \wedge e_4 - [e_2, e_4] \wedge e_1 \wedge e_3 + [e_3, e_4] \wedge e_1 \wedge e_2 \\
&= -e_1 \wedge e_2 \wedge e_4 + e_1 \wedge e_2 \wedge e_3 \\
&= e_1 \wedge e_2 \wedge e_3 - e_1 \wedge e_2 \wedge e_4.
\end{aligned}
$$

Now we may determine $\mathfrak{Im}\partial_k$.

$$
\begin{aligned}
\mathfrak{Im}\partial_1 &= 0, \\
\mathfrak{Im}\partial_2 &=< e_2, e_3 > \\
\mathfrak{Im}\partial_3 &=< -e_1 \wedge e_2, e_1 \wedge e_4, -e_2 \wedge e_3 > \\
&=< e_1 \wedge e_2, e_1 \wedge e_4, e_2 \wedge e_3 >, \\
\mathfrak{Im}\partial_4 &=< e_1 \wedge e_2 \wedge e_3 - e_1 \wedge e_2 \wedge e_4 > .
\end{aligned}
$$

Similarly, we now may determine $\ker \partial_k$.

$$
\begin{aligned}
&\ker \partial_0 = \mathbb{R}, \\
&\ker \partial_1 = g, \\
&\ker \partial_2 =< e_1 \wedge e_2, e_1 \wedge e_4, e_2 \wedge e_3, e_3 \wedge e_4 >, \\
&\ker \partial_3 =< e_1 \wedge e_2 \wedge e_3 - e_1 \wedge e_2 \wedge e_4 >, \\
&\ker \partial_4 = 0.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
H_0^{lie}(\mathfrak{g}) &= \frac{\ker \partial_0}{\mathfrak{Im}\partial_1} \\
&= \frac{\mathbb{R}}{0} \\
&= \mathbb{R}, \\
H_1^{lie}(\mathfrak{g}) &= \frac{\ker \partial_1}{\mathfrak{Im}\partial_2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{g}{< e_2, e_3 >} \\
&=< e_1, e_4 > \\
H_2^{lie}(\mathfrak{g}) &= \frac{\ker \partial_2}{\mathfrak{Im}\partial_3} \\
&= \frac{< e_1 \wedge e_2, e_1 \wedge e_4, e_2 \wedge e_3, e_3 \wedge e_4 >}{< e_1 \wedge e_2, e_1 \wedge e_4, e_2 \wedge e_3 >} \\
&=< e_3 \wedge e_4 >, \\
H_3^{lie}(\mathfrak{g}) &= \frac{\ker \partial_3}{\mathfrak{Im}\partial_4} \\
&= \frac{< e_1 \wedge e_2 \wedge e_3 - e_1 \wedge e_2 \wedge e_4 >}{< e_1 \wedge e_2 \wedge e_3 - e_1 \wedge e_2 \wedge e_4 >} \\
&= 0, \\
H_4^{lie}(\mathfrak{g}) &= \frac{\ker \partial_4}{\mathfrak{Im}\partial_5} \\
&= \frac{0}{0} \\
&= 0.
\end{aligned}
$$

Therefore,

$$
H_k^{lie} = \begin{cases} \mathbb{R} & k = 0 \\ < e_1, e_4 > & k = 1 \\ < e_3 \wedge e_4 > & k = 2, \\ 0 & k \geq 3 \end{cases}
$$

.

Theorem 3.8 *Let* $\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$ *be a Lie algebra isomorphic to* $t_{4,1,\lambda}$, *given by the brackets* $[e_4, e_1], = e_1, [e_4, e_2] = e_2, [e_4, e_3] = \lambda e_3$. *Then,*

$$
H_k^{lie}(\mathfrak{g}) = \begin{cases} \mathbb{R} & k = 0 \\ < e_4 > & k = 1 \\ 0 & k \geq 2 \end{cases}
$$

Proof.

The Chevalley-Eilenberg complex is reduced to

$0 \overset{\partial_0}{\longleftarrow} \mathbb{R} \overset{\partial_1}{\longleftarrow} \mathfrak{g} \overset{\partial_2}{\longleftarrow} \mathfrak{g}^{\wedge^2} \overset{\partial_3}{\longleftarrow} \mathfrak{g}^{\wedge^3} \leftarrow \mathfrak{g} \overset{\partial_4}{\longleftarrow} \mathfrak{g}^{\wedge^4} \leftarrow 0$

$\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$
$\mathfrak{g}^{\wedge 2} = span\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$
$\mathfrak{g}^{\wedge 3} = span\{e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4\}$
$\mathfrak{g}^{\wedge 4} = span\{e_1 \wedge e_2 \wedge e_3 \wedge e_4\}$

$$
\begin{aligned}
\partial_0 &= 0, \\
\partial_1 &= 0, \\
\partial_2(e_1 \wedge e_2) &= [e_1, e_2] \\
&= 0, \\
\partial_2(e_1 \wedge e_3) &= [e_1, e_3] \\
&= 0, \\
\partial_2(e_1 \wedge e_4) &= [e_1, e_4] \\
&= -[e_4, e_1] \\
&= -e_1, \\
\partial_2(e_2 \wedge e_3) &= [e_2, e_3] \\
&= 0, \\
\partial_2(e_2 \wedge e_4) &= [e_2, e_4] \\
&= -[e_4, e_2] \\
&= -e_2, \\
\partial_2(e_3 \wedge e_4) &= [e_3, e_4] \\
&= -[e_4, e_3] \\
&= -\lambda e_3, \\
\partial_3(e_1 \wedge e_2 \wedge e_3) &= [e_1, e_2] \wedge e_3 - [e_1, e_3] \wedge e_2 + [e_2, e_3] \wedge e_1 \\
&= 0, \\
\partial_3(e_1 \wedge e_2 \wedge e_4) &= [e_1, e_2] \wedge e_4 - [e_1, e_4] \wedge e_2 + [e_2, e_4] \wedge e_1 \\
&= [e_4, e_1] \wedge e_2 - [e_4, e_2] \wedge e_1 \\
&= e_1 \wedge e_2 + e_2 \wedge e_2 \\
&= e_1 \wedge e_2, \\
\partial_3(e_1 \wedge e_3 \wedge e_4) &= [e_1, e_3] \wedge e_4 - [e_1, e_4] \wedge e_3 + [e_3, e_4] \wedge e_1 \\
&= [e_4, e_1] \wedge e_3 - [e_4, e_3] \wedge e_1 \\
&= e_1 \wedge e_3 - \lambda e_3 \wedge e_1 \\
&= (\lambda + 1) e_1 \wedge e_3, \\
\partial_3(e_2 \wedge e_3 \wedge e_4) &= [e_2, e_3] \wedge e_4 - [e_2, e_4] \wedge e_3 + [e_3, e_4] \wedge e_2 \\
&= [e_4, e_2] \wedge e_3 - [e_4, e_3] \wedge e_2 \\
&= e_2 \wedge e_3 - \lambda e_3 \wedge e_2 \\
&= (1 + \lambda) e_2 \wedge e_3, \\
\partial_4(e_1 \wedge e_2 \wedge e_3 \wedge e_4) &= [e_1, e_2] \wedge e_3 \wedge e_4 - [e_1, e_3] \wedge e_2 \wedge e_4 + [e_1, e_4] \wedge e_2 \wedge e_3 \\
&\quad + [e_2, e_3] \wedge e_1 \wedge e_4 - [e_2, e_4] \wedge e_1 \wedge e_3 + [e_3, e_4] \wedge e_1 \wedge e_2 \\
&= -[e_4, e_1] \wedge e_2 \wedge e_3 + [e_4, e_2] \wedge e_1 \wedge e_3 - [e_4, e_3] \wedge e_1 \wedge e_2
\end{aligned}
$$

$$
\begin{aligned}
&= -e_1 \wedge e_2 \wedge e_3 - e_1 \wedge e_2 \wedge e_3 - \lambda e_3 \wedge e_1 \wedge e_2 \\
&= -(2+\lambda) e_1 \wedge e_2 \wedge e_3.
\end{aligned}
$$

Now we may determine $\mathfrak{Im}\partial_k$.

$$
\begin{aligned}
\mathfrak{Im}\partial_1 &= 0, \\
\mathfrak{Im}\partial_2 &= < -e_1, -e_2, -\lambda e_3 > \\
&= < e_1, e_2, e_3 >, \\
\mathfrak{Im}\partial_3 &= < e_1 \wedge e_2, (\lambda+1) e_1 \wedge e_3, (1+\lambda) e_2 \wedge e_3 > \\
&= < e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3 >, \\
\mathfrak{Im}\partial_4 &= < -(2+\lambda) e_1 \wedge e_2 \wedge e_3 > \\
&= < e_1 \wedge e_2 \wedge e_3 > .
\end{aligned}
$$

Similarly, we now may determine $\ker \partial_k$.

$$
\begin{aligned}
&\ker \partial_0 = \mathbb{R}, \\
&\ker \partial_1 = g, \\
&\ker \partial_2 = < e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3 >, \\
&\ker \partial_3 = < e_1 \wedge e_2 \wedge e_3 >, \\
&\ker \partial_4 = 0.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
H_0^{lie}(\mathfrak{g}) &= \frac{\ker \partial_0}{\mathfrak{Im}\partial_1} \\
&= \frac{\mathbb{R}}{0} \\
&= \mathbb{R}, \\
H_1^{lie}(\mathfrak{g}) &= \frac{\ker \partial_1}{\mathfrak{Im}\partial_2} \\
&= \frac{g}{< e_1, e_2, e_3 >} \\
&= < e_4 >, \\
H_2^{lie}(\mathfrak{g}) &= \frac{\ker \partial_2}{\mathfrak{Im}\partial_3} \\
&= \frac{< e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3 >}{< e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3 >} \\
&= 0, \\
H_3^{lie}(\mathfrak{g}) &= \frac{\ker \partial_3}{\mathfrak{Im}\partial_4}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{< e_1 \wedge e_2 \wedge e_3 >}{< e_1 \wedge e_2 \wedge e_3 >} \\
&= 0, \\
H_4^{lie}(\mathfrak{g}) &= \frac{\ker \partial_4}{\mathfrak{Im}\partial_5} \\
&= \frac{0}{0} \\
&= 0.
\end{aligned}
$$

Therefore,

$$
H_k^{lie} = \begin{cases} \mathbb{R} & k = 0 \\ < e_4 > & k = 1 \\ 0 & k \geq 2 \end{cases}
$$

.

Theorem 3.9 *Let* $\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$ *be a Lie algebra isomorphic to* $\partial_{4,\lambda}$*, given by the brackets* $[e_4, e_3], = e_3, [e_1, e_2] = e_3, [e_4, e_2] = (1-\lambda)e_2, [e_4, e_1] = \lambda e_1$*. Then,*

$$
H_k^{lie}(\mathfrak{g}) = \begin{cases} \mathbb{R} & k = 0 \\ < e_4 > & k = 1 \\ 0 & k \geq 2 \end{cases}
$$

.

Proof.

The Chevalley-Eilenberg complex is reduced to

$0 \xleftarrow{\partial_0} \mathbb{R} \xleftarrow{\partial_1} \mathfrak{g} \xleftarrow{\partial_2} \mathfrak{g}^{\wedge^2} \xleftarrow{\partial_3} \mathfrak{g}^{\wedge^3} \leftarrow \mathfrak{g} \xleftarrow{\partial_4} \mathfrak{g}^{\wedge^4} \leftarrow 0$

$\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$

$\mathfrak{g}^{\wedge 2} = span\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$

$\mathfrak{g}^{\wedge 3} = span\{e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4\}$

$\mathfrak{g}^{\wedge 4} = span\{e_1 \wedge e_2 \wedge e_3 \wedge e_4\}$

First we must evaluate ∂_k.

$$
\begin{aligned}
\partial_0 &= 0, \\
\partial_1 &= 0, \\
\partial_2(e_1 \wedge e_2) &= [e_1, e_2] \\
&= e_3, \\
\partial_2(e_1 \wedge e_3) &= [e_1, e_3] \\
&= 0, \\
\partial_2(e_1 \wedge e_4) &= [e_1, e_4] \\
&= -[e_4, e_1]
\end{aligned}
$$

$$
\begin{aligned}
&= -\lambda e_1, \\
\partial_2(e_2 \wedge e_3) &= [e_2, e_3] \\
&= 0, \\
\partial_2(e_2 \wedge e_4) &= [e_2, e_4] \\
&= -[e_4, e_2] \\
&= -(1-\lambda)e_2 \\
&= (1+\lambda)e_2, \\
\partial_2(e_3 \wedge e_4) &= [e_3, e_4] \\
&= -[e_4, e_3] \\
&= -e_3, \\
\partial_3(e_1 \wedge e_2 \wedge e_3) &= [e_1, e_2] \wedge e_3 - [e_1, e_3] \wedge e_2 + [e_2, e_3] \wedge e_1 \\
&= 0, \\
\partial_3(e_1 \wedge e_2 \wedge e_4) &= [e_1, e_2] \wedge e_4 - [e_1, e_4] \wedge e_2 + [e_2, e_4] \wedge e_1 \\
&= [e_1, e_2] \wedge e_4 + [e_4, e_1] \wedge e_2 - [e_4, e_2] \wedge e_1 \\
&= e_3 \wedge e_4 + \lambda e_1 \wedge e_2 - (1-\lambda)e_2 \wedge e_1 \\
&= e_1 \wedge e_2 - e_4 \wedge e_3, \\
\partial_3(e_1 \wedge e_3 \wedge e_4) &= [e_1, e_3] \wedge e_4 - [e_1, e_4] \wedge e_3 + [e_3, e_4] \wedge e_1 \\
&= [e_4, e_1] \wedge e_3 - [e_4, e_3] \wedge e_1 \\
&= \lambda e_1 \wedge e_3 - e_3 \wedge e_1 \\
&= (\lambda+1)e_1 \wedge e_3, \\
\partial_3(e_2 \wedge e_3 \wedge e_4) &= [e_2, e_3] \wedge e_4 - [e_2, e_4] \wedge e_3 + [e_3, e_4] \wedge e_2 \\
&= [e_4, e_2] \wedge e_3 - [e_4, e_3] \wedge e_2 \\
&= (1-\lambda)e_2 \wedge e_3 + e_2 \wedge e_3 \\
&= (2-\lambda)e_2 \wedge e_3, \\
\partial_4(e_1 \wedge e_2 \wedge e_3 \wedge e_4) &= [e_1, e_2] \wedge e_3 \wedge e_4 - [e_1, e_3] \wedge e_2 \wedge e_4 + [e_1, e_4] \wedge e_2 \wedge e_3 \\
&+ [e_2, e_3] \wedge e_1 \wedge e_4 - [e_2, e_4] \wedge e_1 \wedge e_3 + [e_3, e_4] \wedge e_1 \wedge e_2 \\
&= -[e_4, e_1] \wedge e_2 \wedge e_3 + [e_4, e_2] \wedge e_1 \wedge e_3 - [e_4, e_3] \wedge e_1 \wedge e_2 \\
&= (1-\lambda)e_2 \wedge e_1 \wedge e_3 - e_3 \wedge e_1 \wedge e_2 \\
&= (\lambda-2)e_1 \wedge e_2 \wedge e_3.
\end{aligned}
$$

Now we may determine $\mathfrak{Im}\partial_k$.

$$
\begin{aligned}
\mathfrak{Im}\partial_1 &= 0, \\
\mathfrak{Im}\partial_2 &=< e_3, -\lambda e_1, (1+\lambda)e_2, -e_3 > \\
&=< e_1, e_2, e_3 >, \\
\mathfrak{Im}\partial_3 &=< e_1 \wedge e_2 + e_3 \wedge e_4, (\lambda+1)e_1 \wedge e_3, (2-\lambda)e_2 \wedge e_3 >
\end{aligned}
$$

$$
\begin{aligned}
&=< e_1 \wedge e_2 - e_4 \wedge e_3, e_1 \wedge e_3, e_2 \wedge e_3 >, \\
\mathfrak{Im}\partial_4 &=< (\lambda - 2) e_1 \wedge e_2 \wedge e_3 > \\
&=< e_1 \wedge e_2 \wedge e_3 > .
\end{aligned}
$$

Similarly, we now may determine $\ker \partial_k$.

$$
\begin{aligned}
&\ker \partial_0 = \mathbb{R}, \\
&\ker \partial_1 = g, \\
&\ker \partial_2 =< e_1 \wedge e_3, e_2 \wedge e_3, e_1 \wedge e_2 - e_4 \wedge e_3 >, \\
&\ker \partial_3 =< e_1 \wedge e_2 \wedge e_3 >, \\
&\ker \partial_4 = 0.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
H_0^{lie}(\mathfrak{g}) &= \frac{\ker \partial_0}{\mathfrak{Im}\partial_1} \\
&= \frac{\mathbb{R}}{0} \\
&= \mathbb{R}, \\
H_1^{lie}(\mathfrak{g}) &= \frac{\ker \partial_1}{\mathfrak{Im}\partial_2} \\
&= \frac{g}{< e_1, e_2, e_3 >} \\
&=< e_4 >, \\
H_2^{lie}(\mathfrak{g}) &= \frac{\ker \partial_2}{\mathfrak{Im}\partial_3} \\
&= \frac{< e_1 \wedge e_3, e_2 \wedge e_3, e_1 \wedge e_2 - e_4 \wedge e_3 >}{< e_1 \wedge e_2 - e_4 \wedge e_3, e_1 \wedge e_3, e_2 \wedge e_3 >} \\
&= 0 \\
H_3^{lie}(\mathfrak{g}) &= \frac{\ker \partial_3}{\mathfrak{Im}\partial_4} \\
&= \frac{< e_1 \wedge e_2 \wedge e_3 >}{< e_1 \wedge e_2 \wedge e_3 >} \\
&= 0, \\
H_4^{lie}(\mathfrak{g}) &= \frac{\ker \partial_4}{\mathfrak{Im}\partial_5} \\
&= \frac{0}{0} \\
&= 0.
\end{aligned}
$$

Therefore,

$$H_k^{lie} = \begin{cases} \mathbb{R} & k = 0 \\ < e_4 > & k = 1 \\ 0 & k \geq 2 \end{cases}$$

.

Theorem 3.10 *Let $\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$ be a Lie algebra isomorphic to $aff(\mathbb{C})$, given by the brackets $[e_4, e_3], = e_3, [e_1, e_2] = e_3, [e_1, e_3] = -e_2, [e_4, e_2] = e_2$. Then,*

$$H_k^{lie}(\mathfrak{g}) = \begin{cases} \mathbb{R} & k = 0 \\ < e_1, e_4 > & k = 1 \\ < e_1 \wedge e_4 > & k = 2 \\ 0 & k \geq 3 \end{cases}$$

.

Proof.

The Chevalley-Eilenberg complex is reduced to

$0 \xleftarrow{\partial_0} \mathbb{R} \xleftarrow{\partial_1} \mathfrak{g} \xleftarrow{\partial_2} \mathfrak{g}^{\wedge^2} \xleftarrow{\partial_3} \mathfrak{g}^{\wedge^3} \leftarrow \mathfrak{g} \xleftarrow{\partial_4} \mathfrak{g}^{\wedge^4} \leftarrow 0$

$\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$

$\mathfrak{g}^{\wedge 2} = span\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$

$\mathfrak{g}^{\wedge 3} = span\{e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4\}$

$\mathfrak{g}^{\wedge 4} = span\{e_1 \wedge e_2 \wedge e_3 \wedge e_4\}$

First we must evaluate ∂_k.

$$\begin{aligned} \partial_0 &= 0, \\ \partial_1 &= 0, \\ \partial_2(e_1 \wedge e_2) &= [e_1, e_2] \\ &= e_3, \\ \partial_2(e_1 \wedge e_3) &= [e_1, e_3] \\ &= -e_2, \\ \partial_2(e_1 \wedge e_4) &= [e_1, e_4] \\ &= 0, \\ \partial_2(e_2 \wedge e_3) &= [e_2, e_3] \\ &= 0, \\ \partial_2(e_2 \wedge e_4) &= [e_2, e_4] \\ &= -[e_4, e_2] \\ &= e_2 \\ \partial_2(e_3 \wedge e_4) &= [e_3, e_4] \end{aligned}$$

$$
\begin{aligned}
&= -[e_4, e_3] \\
&= -e_3, \\
\partial_3(e_1 \wedge e_2 \wedge e_3) &= [e_1, e_2] \wedge e_3 - [e_1, e_3] \wedge e_2 + [e_2, e_3] \wedge e_1 \\
&= 0, \\
\partial_3(e_1 \wedge e_2 \wedge e_4) &= [e_1, e_2] \wedge e_4 - [e_1, e_4] \wedge e_2 + [e_2, e_4] \wedge e_1 \\
&= [e_1, e_2] \wedge e_4 - [e_4, e_2] \wedge e_1 \\
&= e_3 \wedge e_4 - e_2 \wedge e_1 \\
&= e_1 \wedge e_2 - e_4 \wedge e_3, \\
\partial_3(e_1 \wedge e_3 \wedge e_4) &= [e_1, e_3] \wedge e_4 - [e_1, e_4] \wedge e_3 + [e_3, e_4] \wedge e_1 \\
&= [e_1, e_3] \wedge e_4 - [e_4, e_3] \wedge e_1 \\
&= -e_2 \wedge e_4 - e_3 \wedge e_1 \\
&= e_1 \wedge e_3 + e_4 \wedge e_2, \\
\partial_3(e_2 \wedge e_3 \wedge e_4) &= [e_2, e_3] \wedge e_4 - [e_2, e_4] \wedge e_3 + [e_3, e_4] \wedge e_2 \\
&= [e_4, e_2] \wedge e_3 - [e_4, e_3] \wedge e_2 \\
&= e_2 \wedge e_3 + e_2 \wedge e_3 \\
&= 2e_2 \wedge e_3, \\
\partial_4(e_1 \wedge e_2 \wedge e_3 \wedge e_4) &= [e_1, e_2] \wedge e_3 \wedge e_4 - [e_1, e_3] \wedge e_2 \wedge e_4 + [e_1, e_4] \wedge e_2 \wedge e_3 \\
&+ [e_2, e_3] \wedge e_1 \wedge e_4 - [e_2, e_4] \wedge e_1 \wedge e_3 + [e_3, e_4] \wedge e_1 \wedge e_2 \\
&= [e_4, e_2] \wedge e_1 \wedge e_3 - [e_4, e_3] \wedge e_1 \wedge e_2 \\
&= -e_1 \wedge e_2 \wedge e_3 - e_1 \wedge e_2 \wedge e_3 \\
&= -2e_1 \wedge e_2 \wedge e_3.
\end{aligned}
$$

Now we may determine $\mathfrak{Im}\partial_k$.

$$
\begin{aligned}
\mathfrak{Im}\partial_1 &= 0, \\
\mathfrak{Im}\partial_2 &= < e_3, -e_2, e_2, -e_3 > \\
&= < e_2, e_3 >, \\
\mathfrak{Im}\partial_3 &= < e_1 \wedge e_2 - e_4 \wedge e_3, e_1 \wedge e_3 + e_4 \wedge e_2, 2e_2 \wedge e_3 > \\
&= < e_1 \wedge e_2 - e_4 \wedge e_3, e_1 \wedge e_3 + e_4 \wedge e_2, e_2 \wedge e_3 >, \\
\mathfrak{Im}\partial_4 &= < -2e_1 \wedge e_2 \wedge e_3 > \\
&= < e_1 \wedge e_2 \wedge e_3 > .
\end{aligned}
$$

Similarly, we now may determine $\ker \partial_k$.

$$
\begin{aligned}
\ker \partial_0 &= \mathbb{R}, \\
\ker \partial_1 &= g, \\
\ker \partial_2 &= < e_1 \wedge e_4, e_2 \wedge e_3, e_1 \wedge e_2 - e_4 \wedge e_3, e_1 \wedge e_3 + e_4 \wedge e_2 >,
\end{aligned}
$$

$$\ker \partial_3 =< e_1 \wedge e_2 \wedge e_3 >,$$
$$\ker \partial_4 = 0.$$

Thus, we have

$$\begin{aligned}
H_0^{lie}(\mathfrak{g}) &= \frac{\ker \partial_0}{\mathfrak{Im}\partial_1} \\
&= \frac{\mathbb{R}}{0} \\
&= \mathbb{R}, \\
H_1^{lie}(\mathfrak{g}) &= \frac{\ker \partial_1}{\mathfrak{Im}\partial_2} \\
&= \frac{g}{< e_2, e_3 >} \\
&=< e_1, e_4 >, \\
H_2^{lie}(\mathfrak{g}) &= \frac{\ker \partial_2}{\mathfrak{Im}\partial_3} \\
&= \frac{< e_1 \wedge e_4, e_2 \wedge e_3, e_1 \wedge e_2 - e_4 \wedge e_3, e_1 \wedge e_3 + e_4 \wedge e_2 >}{< e_1 \wedge e_2 - e_4 \wedge e_3, e_1 \wedge e_3 + e_4 \wedge e_2, e_2 \wedge e_3 >} \\
&=< e_1 \wedge e_4 >, \\
H_3^{lie}(\mathfrak{g}) &= \frac{\ker \partial_3}{\mathfrak{Im}\partial_4} \\
&= \frac{< e_1 \wedge e_2 \wedge e_3 >}{< e_1 \wedge e_2 \wedge e_3 >} \\
&= 0, \\
H_4^{lie}(\mathfrak{g}) &= \frac{\ker \partial_4}{\mathfrak{Im}\partial_5} \\
&= \frac{0}{0} \\
&= 0.
\end{aligned}$$

Therefore,

$$H_k^{lie}(\mathfrak{g}) = \begin{cases} \mathbb{R} & k = 0 \\ < e_1, e_4 > & k = 1 \\ < e_1 \wedge e_4 > & k = 2 \\ 0 & k \geq 3 \end{cases}$$

.

Theorem 3.11 *Let* $\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$ *be a Lie algebra isomorphic to* ∂_4*,* *given by the brackets* $[e_1, e_2] = e_1, [e_2, e_4] = e_4, [e_1, e_4] = e_3$. *Then,*

$$H_k^{lie}(\mathfrak{g}) = \begin{cases} \mathbb{R} & k = 0 \\ < e_2 > & k = 1 \\ 0 & k = 2, \\ < e_1 \wedge e_3 \wedge e_4 > & k = 3, \\ < e_1 \wedge e_2 \wedge e_3 \wedge e_4 > & k = 4, \\ 0 & k \geq 5 \end{cases}$$

.

Proof.

The Chevalley-Eilenberg complex is reduced to
$0 \xleftarrow{\partial_0} \mathbb{R} \xleftarrow{\partial_1} \mathfrak{g} \xleftarrow{\partial_2} \mathfrak{g}^{\wedge^2} \xleftarrow{\partial_3} \mathfrak{g}^{\wedge^3} \leftarrow \mathfrak{g} \xleftarrow{\partial_4} \mathfrak{g}^{\wedge^4} \leftarrow 0$

$\mathfrak{g} = span\{e_1, e_2, e_3, e_4\}$
$\mathfrak{g}^{\wedge 2} = span\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$
$\mathfrak{g}^{\wedge 3} = span\{e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4\}$
$\mathfrak{g}^{\wedge 4} = span\{e_1 \wedge e_2 \wedge e_3 \wedge e_4\}$
First we must evaluate ∂_k.

$$\begin{aligned} \partial_0 &= 0, \\ \partial_1 &= 0, \\ \partial_2(e_1 \wedge e_2) &= [e_1, e_2] \\ &= e_1, \\ \partial_2(e_1 \wedge e_3) &= [e_1, e_3] \\ &= 0, \\ \partial_2(e_1 \wedge e_4) &= [e_1, e_4] \\ &= e_3, \\ \partial_2(e_2 \wedge e_3) &= [e_2, e_3] \\ &= 0, \\ \partial_2(e_2 \wedge e_4) &= [e_2, e_4] \\ &= e_4, \\ \partial_2(e_3 \wedge e_4) &= [e_3, e_4] \\ &= 0, \\ \partial_3(e_1 \wedge e_2 \wedge e_3) &= [e_1, e_2] \wedge e_3 - [e_1, e_3] \wedge e_2 + [e_2, e_3] \wedge e_1 \\ &= e_1 \wedge e_3, \\ \partial_3(e_1 \wedge e_2 \wedge e_4) &= [e_1, e_2] \wedge e_4 - [e_1, e_4] \wedge e_2 + [e_2, e_4] \wedge e_1 \\ &= e_1 \wedge e_4 + e_2 \wedge e_3 - e_1 \wedge e_4 \end{aligned}$$

$$
\begin{aligned}
&= e_2 \wedge e_3, \\
\partial_3(e_1 \wedge e_3 \wedge e_4) &= [e_1, e_3] \wedge e_4 - [e_1, e_4] \wedge e_3 + [e_3, e_4] \wedge e_1 \\
&= 0, \\
\partial_3(e_2 \wedge e_3 \wedge e_4) &= [e_2, e_3] \wedge e_4 - [e_2, e_4] \wedge e_3 + [e_3, e_4] \wedge e_2 \\
&= e_3 \wedge e_4, \\
\partial_4(e_1 \wedge e_2 \wedge e_3 \wedge e_4) &= [e_1, e_2] \wedge e_3 \wedge e_4 - [e_1, e_3] \wedge e_2 \wedge e_4 + [e_1, e_4] \wedge e_2 \wedge e_3 \\
&+ [e_2, e_3] \wedge e_1 \wedge e_4 - [e_2, e_4] \wedge e_1 \wedge e_3 + [e_3, e_4] \wedge e_1 \wedge e_2 \\
&= e_1 \wedge e_3 \wedge e_4 - e_1 \wedge e_3 \wedge e_4 \\
&= 0.
\end{aligned}
$$

Now we may determine $\mathfrak{Im}\partial_k$.

$$
\begin{aligned}
&\mathfrak{Im}\partial_1 = 0, \\
&\mathfrak{Im}\partial_2 =< e_1, e_3, e_4 > \\
&\mathfrak{Im}\partial_3 =< e_1 \wedge e_3, e_2 \wedge e_3, e_3 \wedge e_4 > \\
&\mathfrak{Im}\partial_4 = 0.
\end{aligned}
$$

Similarly, we now may determine $\ker \partial_k$.

$$
\begin{aligned}
&\ker \partial_0 = \mathbb{R}, \\
&\ker \partial_1 = g, \\
&\ker \partial_2 =< e_1 \wedge e_3, e_2 \wedge e_3, e_3 \wedge e_4 >, \\
&\ker \partial_3 =< e_1 \wedge e_3 \wedge e_4 >, \\
&\ker \partial_4 =< e_1 \wedge e_2 \wedge e_3 \wedge e_4 > .
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
H_0^{lie}(\mathfrak{g}) &= \frac{\ker \partial_0}{\mathfrak{Im}\partial_1} \\
&= \frac{\mathbb{R}}{0} \\
&= \mathbb{R}, \\
H_1^{lie}(\mathfrak{g}) &= \frac{\ker \partial_1}{\mathfrak{Im}\partial_2} \\
&= \frac{g}{< e_1, e_3, e_4 >} \\
&=< e_2 > \\
H_2^{lie}(\mathfrak{g}) &= \frac{\ker \partial_2}{\mathfrak{Im}\partial_3}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{< e_1 \wedge e_3, e_2 \wedge e_3, e_3 \wedge e_4 >}{< e_1 \wedge e_3, e_2 \wedge e_3, e_3 \wedge e_4 >} \\
&= 0, \\
H_3^{lie}(\mathfrak{g}) &= \frac{\ker \partial_3}{\mathfrak{Im}\partial_4} \\
&= \frac{< e_1 \wedge e_3 \wedge e_4 >}{0} \\
&=< e_1 \wedge e_3 \wedge e_4 >, \\
H_4^{lie}(\mathfrak{g}) &= \frac{\ker \partial_4}{\mathfrak{Im}\partial_5} \\
&= \frac{< e_1 \wedge e_2 \wedge e_3 \wedge e_4 >}{0} \\
&=< e_1(\mathfrak{g}) \wedge e_2 \wedge e_3 \wedge e_4 > .
\end{aligned}
$$

Therefore,

$$
H_k^{lie}(\mathfrak{g}) = \begin{cases} \mathbb{R} & k = 0 \\ < e_2 > & k = 1 \\ 0 & k = 2, \\ < e_1 \wedge e_3 \wedge e_4 > & k = 3, \\ < e_1 \wedge e_2 \wedge e_3 \wedge e_4 > & k = 4, \\ 0 & k \geq 5. \end{cases}
$$

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