SEQUENCES OF SODDY CIRCLES

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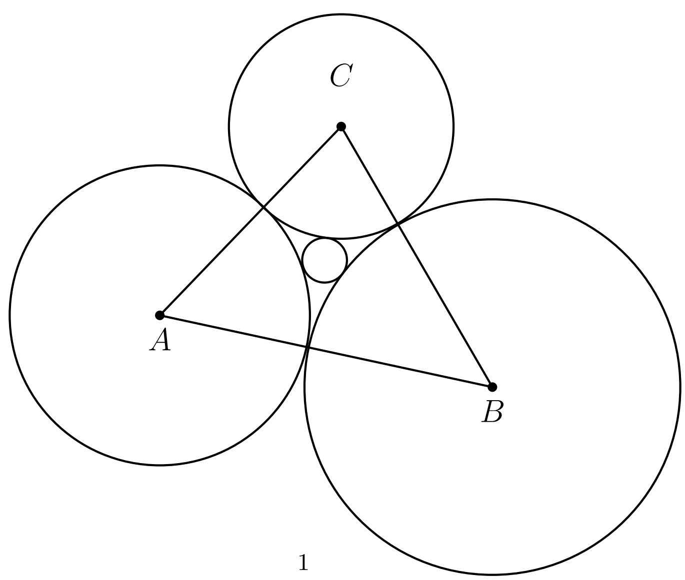
1. Introduction

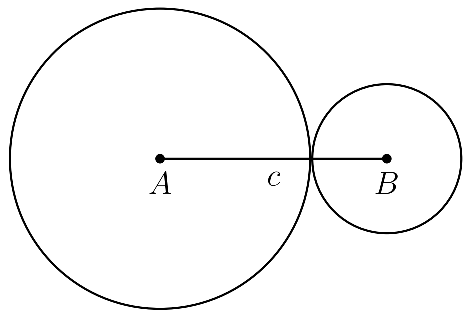
Let there be a triangle $\triangle ABC$ and let there be a circle centered at each vertex of $\triangle ABC$, $\mathcal{C}_A, \mathcal{C}_B$, and $\mathcal{C}_C$. When these three circles are externally tangent, there exists fourth externally tangent circle called the inner Soddy circle. If we fixed two of the original three circles, we can use that Soddy circle to find another inner Soddy circle and we can do this indefinitely. We are interested in what happens to the radii of the inner Soddy circles if we let n go to infinity and similarly what would happen to the ratio the radii.

Research on how to relate the center of the Soddy circle with the other three circles and other important points in geometry is not new. Oldknow uses the incentre I and the centroid G to defined the Soddy center S as the following $S = I + G$ and then uses that to define the Soddy lines. Eppstein used Soddy centers to relates a point of tangency to the incentre I and the Gregonne point Ge. We are interested in how the centers of the inner Soddy circles relate to the other three circles in terms of their radii and the curve that passes through them.

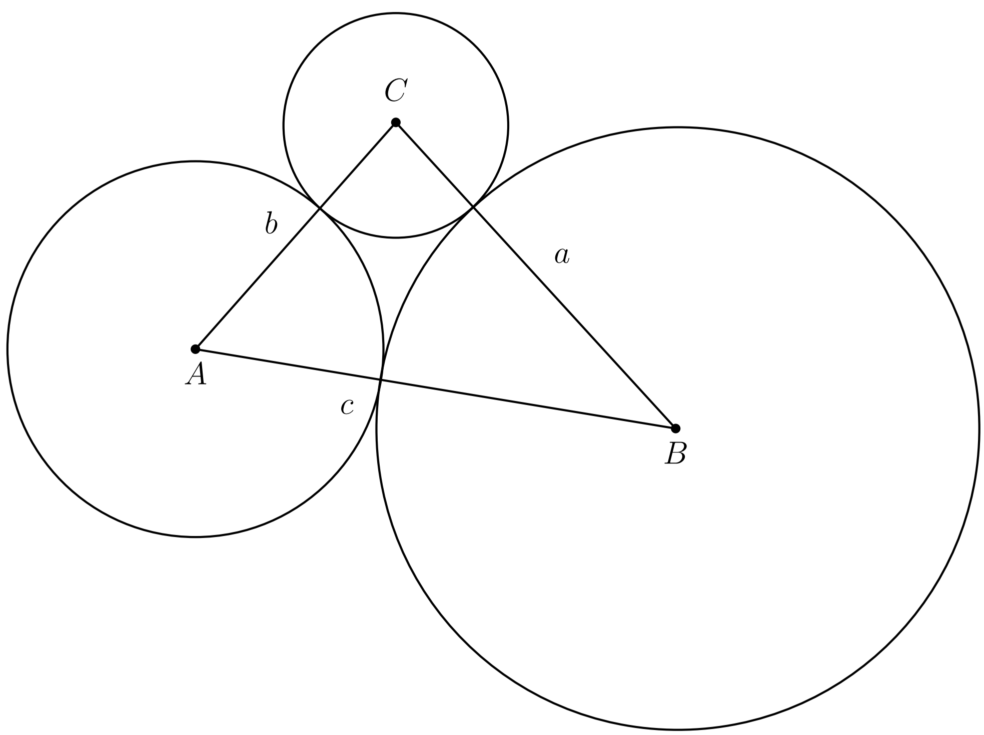


2. Mutually Tangent Circles

Let AB be a segment of length c. Let $\mathcal{C}_A$ and $\mathcal{C}_B$ be externally tangent circles centered at A and B, respectively.



Then $\mathcal{C}_A$ and $\mathcal{C}_B$ are tangent if $c = r_A + r_B$. Because of this linear condition, r_A can be written in terms of r_B and similarly r_B can be written in terms of r_A. Therefore there are infinitely many pairs of externally tangent circles that satisfy this condition. Now let A, B, and C be three noncollinear points. AB is of length c, BC is of length a, and CA is of length b. Let $\mathcal{C}_A$, $\mathcal{C}_B$, and $\mathcal{C}_C$ be circles centered at A, B and C respectively.



Then $\mathcal{C}_A$, $\mathcal{C}_B$, and $\mathcal{C}_C$ are mutually externally tangent if the following linear conditions hold.

$$a = r_B + r_C$$
$$b = r_A + r_C$$
$$c = r_A + r_B$$

Solve this system of linear equations,

$$r_A = \frac{1}{2}(-a + b + c)$$

$$r_B = \frac{1}{2}(a - b + c)$$

$$r_C = \frac{1}{2}(a + b - c).$$

Let s be the semiperimeter of $\triangle ABC$ defined as $s = \frac{1}{2}(a + b + c)$, then

$$r_A = s - a$$

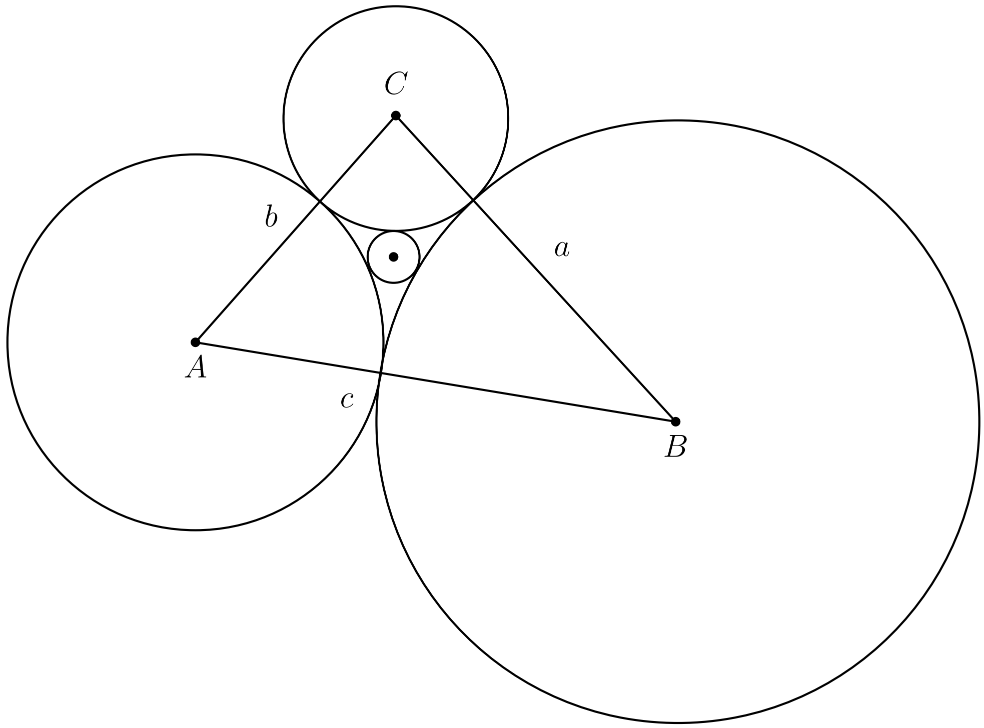
$$r_B = s - b$$

$$r_C = s - c$$

Thus we have a unique solution and therefore only one configuration such that three circles are externally tangent.

3. Soddy Circles and the "Four Coin" Problem

Within the three mutually externally tangent circles described in Section 2, there exists a fourth circle that is externally tangent to all three.



This circle is the inner Soddy circle, $\mathcal{C}_S$, named after Fredrick Soddy. Soddy was an English chemist who was interested in solving the "Four Coins" problem, which asked, given three mutually externally tangent circles and a fourth circle within those three, how does the fourth radius relate to those of the other three circles and where is the fourth center with respect to the other three centers? While he did not provide an answer for the second of the two questions, Soddy did give an answer for the first question in the form of a poem called "The Kiss Precise".

Four circles to the kissing come,
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have a minus sign,
The sum of the squares of all four bend
Is half the square of their sum.

Although Soddy is the one given credit for this revelation, Rene Descartes had already discovered the radii of the Soddy circles some two hundred years earlier[1].

Theorem 3.1 (Descartes' Circle Theorem)**.** *Given four mutually tangent circles, define the curvatures of each circle as* $\varepsilon_A = \frac{1}{r_A}$*,* $\varepsilon_B = \frac{1}{r_B}$*,* $\varepsilon_C = \frac{1}{r_C}$*, and* $\varepsilon_S = \frac{1}{r_S}$*. Then the formula relating the four radii is*

$$2(\varepsilon_A^2 + \varepsilon_B^2 + \varepsilon_C^2 + \varepsilon_S^2) = (\varepsilon_A + \varepsilon_B + \varepsilon_C + \varepsilon_S)^2.$$

Using Descartes' Circle Theorem, we can solve for the radius of the inner Soddy circle.

$$2(\varepsilon_A^2 + \varepsilon_B^2 + \varepsilon_C^2 + \varepsilon_S^2) = (\varepsilon_A + \varepsilon_B + \varepsilon_C + \varepsilon_S)^2$$

We distribute and rearrange until we get,

$$\varepsilon_A^2+\varepsilon_B^2+\varepsilon_C^2+\varepsilon_S^2+2\varepsilon_A\varepsilon_B+2\varepsilon_A\varepsilon_C+2\varepsilon_B\varepsilon_C-2\varepsilon_A\varepsilon_S-2\varepsilon_B\varepsilon_S-2\varepsilon_C\varepsilon_S = 4(\varepsilon_A\varepsilon_B+2\varepsilon_A\varepsilon_C+\varepsilon_B\varepsilon_C)$$

Taking the square root of both sides we get,

$$\varepsilon_S - \varepsilon_A - \varepsilon_B - \varepsilon_C = \pm 2\sqrt{\varepsilon_A\varepsilon_B + \varepsilon_A\varepsilon_C + \varepsilon_B\varepsilon_C}$$

Solving for ε_S,

$$\varepsilon_S = \varepsilon_A + \varepsilon_B + \varepsilon_C \pm 2\sqrt{\varepsilon_A\varepsilon_B + \varepsilon_A\varepsilon_C + \varepsilon_B\varepsilon_C}$$

Replacing the ε with the radii reciprocals,

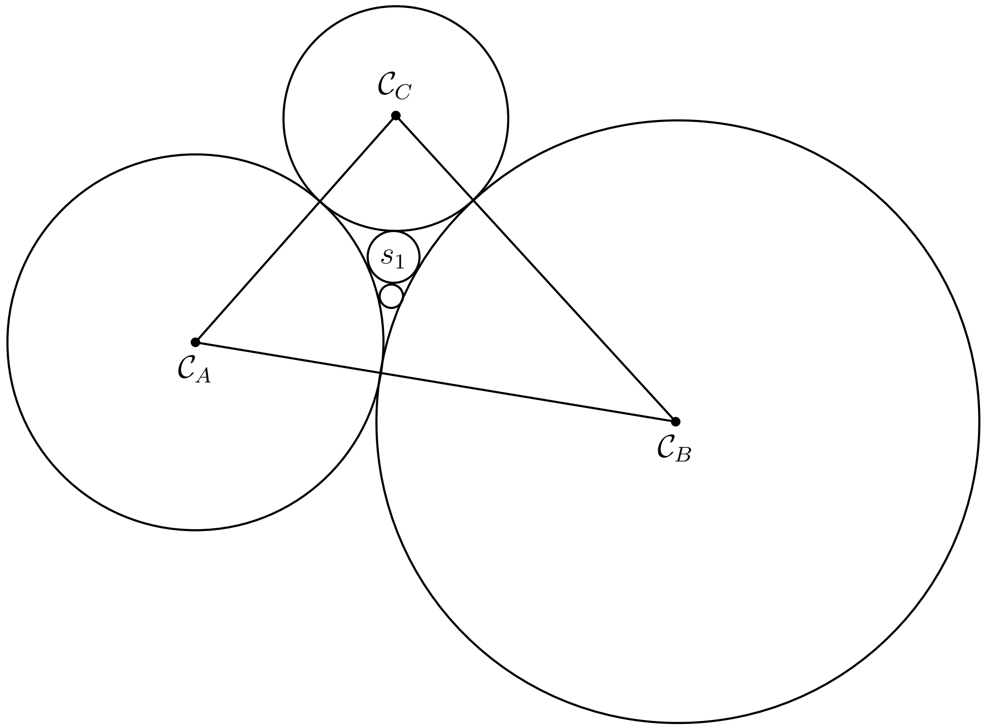
$$\frac{1}{r_S} = \frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C} \pm 2\sqrt{\frac{1}{r_Ar_B} + \frac{1}{r_Ar_C} + \frac{1}{r_Br_C}}$$

Therefore

$$r_s = \frac{r_Ar_Br_C}{r_Ar_B + r_Ar_C + r_Br_C \pm 2\sqrt{r_Ar_Br_C(r_A + r_B + r_C)}}$$

4. Sequences of Soddy Circles

If we fix $\mathcal{C}_A$ and $\mathcal{C}_B$, we can create a new inner Soddy circle by letting the Soddy circle s_1 be the new $\mathcal{C}_C$ to find a new inner Soddy circle s_2.



We can continuously do this for s_n and write it as the sequence

$$s_{n+1} = \frac{r_A r_B s_n}{r_A s_n + r_B s_n + r_A r_B + 2\sqrt{r_A r_B s_n (r_A + r_B + s_n)}}$$

We only consider the $+$ of the $\pm$ since that correlates to the inner Soddy circle. The $-$ is still a viable answer, but it concerns the outer Soddy circle, which we are not considering in this paper.

Using Maple, we made the conjecture that the

$$\lim_{n\to\infty} s_n = 0.$$

In Maple let $a = r_A, b = r_B$, and $c = r_C$. Then to find the limit of s_n we use the following code:

$s := c;$

for n from 1 to 1000 do

$$s := \frac{a \cdot b \cdot s}{a \cdot s + b \cdot s + a \cdot b + 2\sqrt{a \cdot b \cdot s(a + b + s)}};$$

end do

where last four iterations of the code give the following:

s:=0.000001984836446

s:=0.000001980883827
s:=0.000001976943008
s:=0.000001973013941.

As we can see the limit of $\{s_n\}$ converges to 0 slowly. This is interesting because if we take three different sized coins, the Soddy circle is much smaller than the other three circles.

Geometrically, we can see that the inner Soddy circle's radius is decreasing and since s_n is always positive we know that $\{s_n\}$ is bounded below by 0 therefore $\{s_n\}$ converges by the Monotone Convergence Theorem. Denote the limit of $\{s_n\}$ by L. Let $s_{n+1} = f(s_n)$ where f is continuous. So if $\{s_n\}$ converges to L, $f(s_n)$ will converge to $f(L)$. Thus $L = f(L)$.

Solving for L,

$$L = \frac{r_A r_B L}{r_A L + r_B L + R_A r_B + 2\sqrt{r_A r_B L(r_A + r_B + L)}}$$

$$r_A L^2 + r_B L^2 + r_A r_B L + 2L\sqrt{r_A r_B L(r_A + r_B + L)} - r_A r_B L = 0$$

$$L(r_A L + r_B L + 2\sqrt{r_A r_B L(r_A + r_B + L)}) = 0.$$

Then

$$L = 0 \text{ or } r_A L + r_B L + 2\sqrt{r_A r_B L(r_A + r_B + L)} = 0.$$

Since $r_A > 0$ and $r_B > 0$, the only way for $r_A L + r_B L + 2\sqrt{r_A r_B L(r_A + r_B + L)}$ to equal 0 is if $L = 0$. Thus $\{s_n\}$ converges to 0.

We can write the ratio of inner Soddy radii as

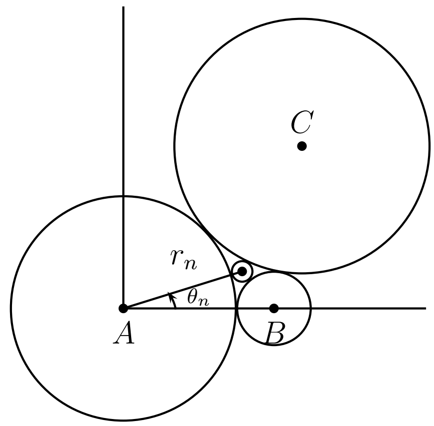
$$\frac{s_{n+1}}{s_n} = \frac{r_A r_B}{r_A s_n + r_B s_n + r_A r_B + 2\sqrt{r_A r_B s_n(r_A + r_B + s_n)}}.$$

Taking the limit of this ratio, we already proved that $\{s_n\}$ converges to 0, therefore all the denominator except for $r_A r_B$ converges to 0 and we can rewrite the limit as

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \frac{r_A r_B}{r_A r_B} = 1.$$

5. Curve Through Soddy Centers

Let A be a fixed point at the origin and B be a fixed point at $(c, 0)$. Now choose any point C, which will determine the configuration of the circles $\mathcal{C}_A, \mathcal{C}_B$, and $\mathcal{C}_C$. The distance between A and B is $r_A + r_B$, the distance between A and C is $r_A + r_C$, and the distance between B and C is $r_B + r_C$.



Using polar coordinates, we can find the Cartesian coordinates for the n^{th} inner Soddy circle. First we get $r_n = r_A + s_n$, where s_n is the radius of the n^{th} inner Soddy circle. Using the law of cosines we get

$$\cos(\theta_n) = \frac{(r_A + s_n)^2 + (r_A + r_B)^2 - (r_B + s_n)^2}{2(r_A + s_n)(r_A + r_B)}.$$

Now we find

$$\begin{aligned} x_n &= r_n \cos(\theta_n) \\ &= \frac{(r_A + s_n)^2 + (r_A + r_B)^2 - (r_B + s_n)^2}{2(r_A + r_B)} \\ &= r_A + s_n \frac{r_A - r_B}{r_A + r_B}. \end{aligned}$$

To find y_n notice that $x_n^2 + y_n^2 = r_n^2$. Therefore

$$\begin{aligned} y_n &= \sqrt{r_n^2 - x_n^2} \\ &= \sqrt{(r_A + s_n)^2 - (r_A + s_n \frac{r_A - r_B}{r_A + r_B})^2} \\ &= \sqrt{\frac{4 r_A r_B s_n (s_n + 1)}{(r_A + r_B)^2}} \\ &= \frac{2\sqrt{r_A r_B s_n (s_n + 1)}}{(r_A + r_B)}. \end{aligned}$$

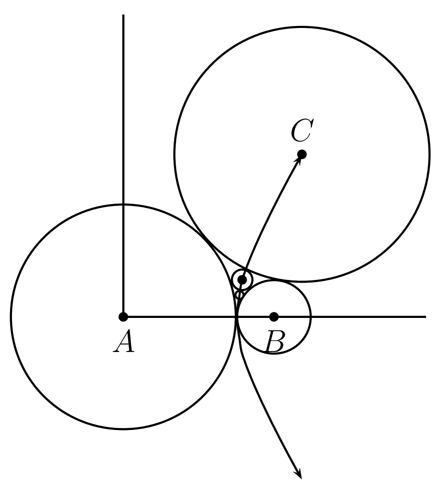
If we let $r_A = r_B$, then $x_n = r_A$ and $y_n = \sqrt{s_n(s_n + 1)}$. In this case, all the Soddy centers lie on the line $x_n = r_A$. Now let $r_A = \alpha r_B$. Then

$$x_n = r_A + s_n \left(\frac{\alpha - 1}{\alpha + 1} \right).$$

Solving for s_n and replacing that in y_n we get the curve

$$y_n = \frac{2\sqrt{\alpha(x_n - r_A)\left(\frac{\alpha+1}{\alpha-1}\right)\left((x_n - r_A)\left(\frac{\alpha+1}{\alpha-1}\right)+1\right)}}{\alpha+1}.$$

When this function is plotted with sample r_a's and r_B's, it looks to be a hyperbolic function. To determine which part of the hyperbolic function is the curve through the Soddy centers we must consider the geometric structure. For instance if $r_A > r_B$, the original inner Soddy center would lie closer to $\mathcal{C}_B$ so we consider the part of the function curving around $\mathcal{C}_B$ as the line through the Soddy circles.



6. Conclusion

After examining the sequence of Soddy circles we can conclude that the limit of the sequence approaches 0 and that the limit of their ratios approaches 1. We can also conclude that there is a line or curve passing through the Soddy centers depending on the relationship of the fixed circles' radii. The next topic to explore is Soddy spheres in a similar way that we explored Soddy circles in this paper. We would look for the convergence of the sequence of Soddy sphere radii and their ratios and what curve would pass through the centers. We could also consider reversing the sequence of Soddy circles and determine what would be the maximum angles of the triangle formed by two fixed circles and the inner Soddy circles before the Soddy circle becomes internally tangent. We could also look at what happens to the angles of the triangle as we go through each iteration of the sequence.

References

[1] Coxeter, H. S. M. *Introduction to Geometry*. New York: Wiley, 1969. Print.

[2] Eppstein, David. “Tangent Spheres and Triangle Centers.” *The American Mathematical Monthly* 108.1 (2001): 63-66. Web.

[3] Oldknow, Adrian. ”The Euler-Gergonne-Soddy Triangle of a Triangle.” *The American Mathematical Monthly* 103.4 (1996):319-29. Web.