Conceptual Understanding of Mathematics in America

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Abstract

Being able to teach and understand mathematics conceptually is essential for mathematics teachers. However, even though American teachers spend more time in mathematics classrooms as learners of mathematics than their Chinese counterparts, Chinese students have shown better understanding of the concepts they are taught and are better able to apply their knowledge than American students. Often the reason cited for this is how mathematics is presented to students in America. Mathematics in America is taught on a procedural level that lacks the depth needed to fully rationalize ideas and concepts. I believe that creating an environment for conceptual understanding, critical thinking, and a more productive disposition for the subject in this country is not only beneficial but imperative for the American education of mathematics.

The Misconceptions in America

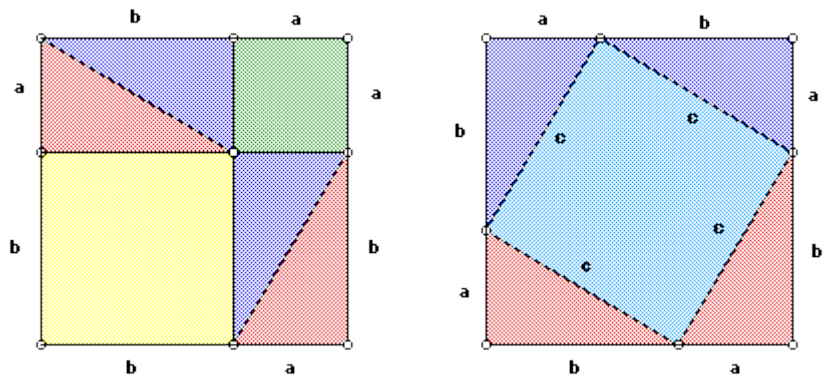
Mathematics has commonly been referred to as the language of nature. Therefore it would be easy to infer that mathematics is formed and connected in a very intuitive way; that mathematics can be discovered and even created through the process of exploration and ingenuity. However, through my observations I believe that a vast majority of Americans cannot make simple conjectures about mathematical phenomena. While most people use different forms of math in their everyday lives without even noticing, if one were to bring up the idea of "doing mathematics" this would bring a cringe to the faces of many. I believe that the process of doing mathematics strikes fear in the hearts of so many because of the methods in which they were taught math growing up in their school systems. Mathematics has been deemed intimidating and cruel to Americans because it appears to be only procedural and this is the sad misconception. There's a beauty in mathematics that more people deserve to see and appreciate.

Growing up in the public school system in Georgia, I was always deemed to be good in math. The reason was because I could do quick arithmetic and happened to be able to see and decipher patterns in math without much guidance. My ability to recognize these patterns and make connections in mathematics is truly the reason I exceled. I had a mindset that thrived in mathematics and therefore I enjoyed the process of understanding it. What I believe to be the case now is that with a few exceptions almost anyone can learn to understand and appreciate mathematics in the same light as I do. Math is taught in America today in a manner that lacks the insight needed to distinguish these patterns and connections for most students. I fortunately stumbled into a passion for math, and I now I want to show how I believe it can be fostered in others. If we as teachers strip down these procedures and start asking the question "Why?" more often then we can break through the walls for a lot of students while building a stronger foundation in what is means to do mathematics. A stronger foundation will yield

more confidence with students in mathematics, which will in time breed a more productive disposition for pushing the envelope with ideas and understanding. The best teacher any student is ever going to have is themselves and mathematics is the greatest subject for exploration and critical thinking. All it takes is a mindset; one that should be created and fostered in America's schools.

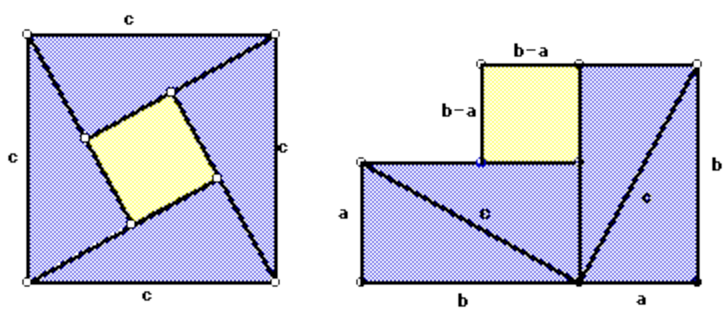
Having a strong foundation in math does not mean that a person is drilled with procedures and formulas at a young age until it is permanently engraved in their memory. A strong foundation means that students are put in an environment as early as possible that gives them the opportunities to discover, build, and connect mathematical concepts for themselves. One great characteristic of math that goes highly unrecognized is that there are almost always multiple ways to arrive at correct solutions for mathematical problems. Therefore, there are always different perspectives in doing mathematics that are all beneficial in different scenarios with different people. Thus, lecturing students into believing that there is always a specific formula or procedure for arriving at answers handicaps students' appreciation for this aspect. Formulas and procedures will always be a great aid to mathematical thought, but they are not a crutch from which to build upon. For example, there ninety-eight different proofs for what we call the Pythagorean Theorem (Bogomolny). Therefore, there are at least ninety-eight unique ways people have argued that when you have a right triangle the sum of the squares of its legs is equivalent to the square of the hypotenuse. Even greater than that, centuries before the Greek named Pythagoras proved this idea, other ancient civilizations before him understood this relationship independently of one another. Therefore we have this one idea about the relationship of a hypotenuse with its legs that has been recognized and theorized from so many different perspectives. While some may have come to discover and verify this idea in a very similar fashion, they are all different. This is just one well-documented example of how mathematical thought can come from many different perspectives. If this is the case for most of the mathematics learned in schools, then why is it taught as

though there is always only one path for arriving at a solution? Here are some examples of how ancient people discovered the Pythagorean Theorem:



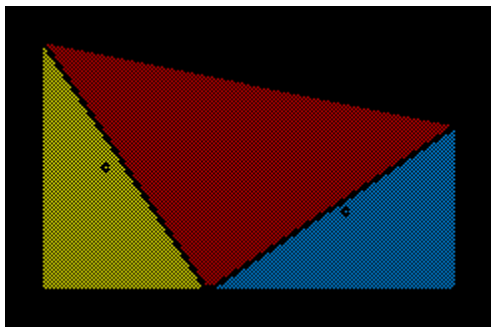
These two squares are congruent to one another. The square on the left has an area of $(A+B)^2$. The area of the square on the right is $4\left(\frac{1}{2}AB\right)+C^2$. If we set these two areas equal to one another,

$$(A+B)^2 = 4\left(\frac{1}{2}AB\right)+C^2$$
$$A^2+2AB+B^2 = 2AB+C^2$$
$$A^2+B^2 = C^2$$



This proof has 4 congruent traingles rotated together and it makes a square with area C^2. If we let the other two sides be A and B, then each one of the triangles has a area of $\frac{1}{2}AB$. Since there are 4 triangles then the total area for them is $2AB$. The middle square has an area of $(A-B)^2$, which in turn makes up the entire area of the larger square when added to the area of the 4 triangles. So,

$$C^2 = 2AB+(A-B)^2$$
$$C^2 = 2AB+A^2-2AB+B^2$$
$$C^2 = A^2+B^2$$



Let the two sides of the two congruent triangles be A and B. The area of this whole trapezoid would be $\frac{1}{2}(A^2 + 2AB + B^2)$. The would in turn be equivalent to the sum of the areas of the three triangles. The triangle on the left would have an area of $\frac{1}{2}BA$. The middle triangle would have $\frac{1}{2}C^2$. The right hand triangle would be $\frac{1}{2}AB$. Setting this equal to the trapezoid,

$$\frac{1}{2}(A^2 + 2AB + B^2) = \frac{1}{2}(AB + C^2 + AB)$$
$$A^2 + 2AB + B^2 = C^2 + 2AB$$
$$A^2 + B^2 = C^2$$

Procedural Mathematics with Subtraction

When math is taught so procedurally, I have come to discover that even the most basic ideas can be wrought with misconceptions or best case memorized as isolated bits of information that have no network of interrelated ideas to build unto. If students have these misconceptions on a fundamental level then they will have nothing with which to build from. For example, Liping Ma (1999) discussed the process of teaching second grade students subtraction. To subtract $75 - 12$ is taught to students in a step by step process. First, you subtract $5 - 2$ to get 3 and then $7 - 1$ to get 6 to receive your answer of 63. However, when students see an equation such as $62 - 49$ they are taught a procedure called "borrowing." When students attempt to subtract $2 - 9$ they are often told that they cannot take 9 from 2 and therefore you have to "borrow a 1" from the 6 in 62 to make the 2 then magically become 12. Most students would then have something written like this:

$$\begin{array}{r} {}^{5}\cancel{6}\ {}^{1}2 \\ \underline{-49} \\ 13 \end{array}$$

This is a basic misconception of subtraction that is taught regularly in America. First, it is a misconception that you cannot take 9 from 2, and teaching students that this cannot be done can lead to more complications when these students start trying to understand operations with integers. Secondly, we teach that we are "borrowing a 1" which is not true. What we are actually doing is "regrouping one 10 as ten ones." If one were to actually break this procedure down, it would look like this:

$$62 - 49 = (60 + 2) - (40 + 9) = \big((50 + 10) + 2\big) - (40 + 9) = (50 + 12) - (40 + 9)$$
$$= (50 - 40) + (12 - 9) = 10 + 3 = 13.$$

This idea may be deemed “basic” and “simple” but it contains the rudiments of advanced mathematical concepts. If we teach these ideas without the thorough process needed, we can hinder students’ abilities to understand greater mathematical concepts in their futures.

China vs. America

Research suggests that American math teachers do have more hours of extensive formal schooling than Chinese teachers. Chinese teachers have 11 to 12 years of formal schooling in contrast to American teachers having received between 16 to 18 years of school with a bachelor's degree and often additional schooling after that (Ma, 1999). Paradoxically, Chinese students have outperformed American students in comparisons of mathematical competency for over 30 years. The reason I believe that Chinese students outperform their American counterparts is due to the fact that while American teachers have been taught mathematics formally for a longer duration of time than Chinese teachers, the Chinese teachers themselves have specialized in a profound understanding of fundamental mathematics. The Chinese teachers understand what they are teaching to their students beyond just computational comprehension. They understand their craft in such a conceptual manner that they can present mathematics in multiple perspectives that not only lets students build mathematical thought in their own way of understanding but also creates an environment for critical thinking.

So, an obvious question comes as a result: How and what are the Chinese teachers doing differently than American teachers that produce this level of understanding? Liping Ma (1999) argued that the key to being an excellent teacher in math is for the teacher to have a profound understanding of fundamental mathematics (PUFM). She writes,

> Profound understanding of fundamental mathematics (PUFM) is more than a sound conceptual understanding of elementary mathematics – it is the awareness of the conceptual structure and basic attitudes of mathematics inherent in elementary mathematics and the ability to provide a foundation for that conceptual structure and instill those basic attitudes in students. A profound understanding of mathematics has *breadth*, *depth*, and *thoroughness.* Breadth of understanding is the capacity to connect a topic with topics of similar or less conceptual power. Depth of understanding is the capacity to connect a topic with those of greater conceptual power. Thoroughness is the capacity to connect all topics. The teaching of a teacher with PUFM has connectedness, promotes multiple approaches to solving a given problem, revisits and reinforces basic ideas, and has longitudinal coherence (Ma, 1999, p. 57).

This level of understanding is not in most Chinese teachers when they first become teachers. Most Chinese teachers that developed what Ma considered being PUFM did so during their teaching careers. Teachers in China have a supporting cast around them that promotes not simply understanding how to *do* mathematics but rather how to *teach* it. Not only do their peers promote understanding of how to teach these concepts but also how to teach each concept in multiple perspectives to give students the best possible chance to comprehend each idea on a thorough conceptual level. They teach with the goal of creating links to the knowledge and the students. This is a mindset based on inspiring and motivating students to learn and build ideas. I also believe this is a philosophy that creates interest in the subject which is imperative. Chinese teachers devote time each week discussing and reflecting on their lesson plans with their colleagues to attempt to perfect their craft. They work together to achieve a PUFM collectively that not only promotes reflection but encourages teachers to continue to learn and build upon their understandings.

So, in China, new teachers are introduced to this setting that encourages them to be the best possible teacher they can. Chinese teachers are asked to take a step back from their current teachings and reflect on their recent failures and achievements and discuss their findings with their peers. They are given a forum with which to build conceptual understanding, confidence, new perspectives, higher ideas, and interest. I believe this environment overflows into the classroom and is what truly separates the two countries' respective abilities to teach conceptual mathematics. The reason being is because conceptual understanding has to begin with the teacher. Since Chinese teachers have this support group, they can advance their understanding of material continually through their careers. The more perspectives a teacher is shown for a given concept solidifies their own understanding and also allows for the teacher to better approach students how may have a different understanding. In America, most teachers are isolated from their peers when it comes to furthering their own respective understanding of mathematics. Teachers may ask questions and get help from other teachers occasionally in America

but they are not given this weekly time period to increase their knowledge as a whole. This is not a fault of the teachers, but rather of the system in which they work. Teachers need a supporting cast to be able to better themselves. Teachers in America do not have much free time to converse these ideas with each other because they are already under the stresses of teaching to standards individually. Standards in recent past have been setup to where teachers are almost competing with each other to meet standards; therefore creating more isolation among peers. There is a lack of a true teaching community in America and instead of pushing for a merit system or any other system that makes teachers compete, America should refocus its efforts to create a teaching culture. If America could allow for teachers to come together with the efforts of bettering each other collectively, then that could potentially revolutionize teaching conceptually within itself. The benefits also extend further than simply communication. Being able to trust and depend on the help and advice of peers is stress-relieving, confidence building, and holds teachers accountable to better themselves. Teaching students at any level carries a heavy burden on teachers, and they deserve the support from one another.

The reason I believe this will increase the teaching of conceptual mathematics is because this forum creates a passion in which to do so. If teaching conceptually truly increases students' abilities to retain and reproduce mathematical concepts then this system would definitely show its benefits. The reason being is that all it would take is one teacher to show an effective conceptual strategy as to how they presented a topic to their classroom and others would recognize its benefits. Teaching mathematics is centered on how topics are presented to students by the teacher; not every method a teacher takes to present concepts to students is going to be the most effective. Therefore, if given the ability to reflect and revise their lessons with their peers, teachers can perfect their lessons and hone their craft. Teaching conceptually then continually increases in effectiveness with experience. Plus, it creates a positive beginning for any new teacher coming into a school system. With all of the stresses

placed on teachers in America, a great support group would benefit the teaching of mathematics incredibly.

Breaking Down Procedural and Conceptual Knowledge

So far I have discussed how China achieves a greater ability to teach conceptual mathematics. However I have not shown what it means to teach conceptual understanding and the differences with how Americans are taught today. For this I will give examples and discuss the differences in the two philosophies. We will start with more basic ideas and work to advanced concepts.

So, for our first example, what does it truly mean to multiply? I remember very well being taught multiplication in third grade, and this is ironically when my passion for mathematics began. The way my class was taught multiplication was by giving us the definition that multiplication is "repeated addition; that if you have $A * B$ then that means you have a number A being added to itself B amount of times." That might sound reasonable now, but at the time it just seemed like an unnecessary hassle. My argument was that if it's just repeated addition then why don't we just continue doing addition and disregard multiplication all together? I convinced myself that it must just be "shorthand addition". Then we were given our multiplication tables and instructed that every week we would be quizzed on a table for each individual number from 2 to 12. This process was nothing more than the straight memorization of numbers. My ability to notice patterns led me to some helpful ideas. I saw that multiplication was commutative, meaning if I knew what $A * B$ was then I also knew $B * A$; therefore for every next number we had to memorize, I already knew the answers for the table up to that number (particularly, If I knew my multiplication tables for 2,3,4,5 then I also know my table for 6 all the way through $6 * 5$). I noticed patterns with numbers 2,4,5,9, and 11. I was just lucky to have the some desire to want to see these patterns, which wasn't the case for everyone in my class. For the ones who struggled, they simply got drilled repeatedly until they finally memorized their tables. One of the hardest numbers to

memorize was the table for 7 because it didn't really seem to follow any obvious pattern. Here's a more conceptual understanding of how to arrive at answers for 7 that I wish I had been shown all those years ago:

$$(7 * 8) = (5 + 2) * 8 = (5 * 8) + (2 * 8) = 40 + 16 = 56$$

The reason why this is simpler than multiplying by 7 directly is because it is normally considered to be easy computation to multiply a number by 5 or 2. Since $7 = 5 + 2$, we can just break it up. This strategy can also work for a lot of scenarios that we will give examples for later.

Later on in my learning of multiplication we were taught how to multiply multi-digit numbers together. This was the fun process of magically moving numbers. Consider $56 * 84$. Most students are instructed to set up the equation as follows:

$$\begin{array}{r} 56 \\ \underline{*\,84} \\ 224 \\ \underline{+448} \\ 4704 \end{array}$$

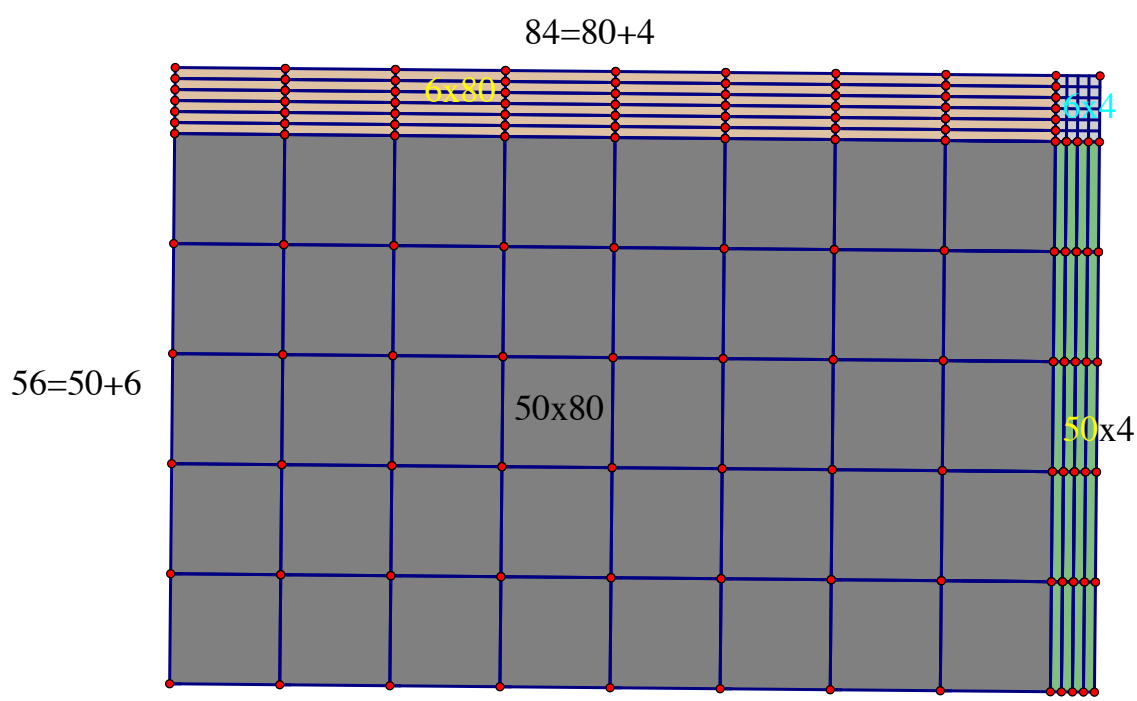
Our instructions were to multiply $6 * 4$ to get 24. Then you "write the 4 under the line and carry the 2. Next you do $5 * 4$ to get 20 and then "add the two" to get the final number of 224. You do this same method for $6 * 8$ and then $5 * 8$ and add a "carried over" 4 to get 448. However, when you write 448 you "skip a space". If you had a three digit number or higher than you would continually skip one more space for every digit you multiplied. The most common mistake students make, a mistake that I continually saw even in college, is this answer:

$$\begin{array}{r} 56 \\ \underline{*\,84} \\ 224 \\ \underline{+448} \\ 692 \end{array}$$

The problem is that the student forgot to “skip a space”. The reason why this is such a common mistake is because it is such an easily forgettable procedure. A lot of the times there was no logic provided to students as to why we skip spaces. I was told to multiply $56 * 8$ and leave and space but that is conceptually wrong. This is a procedure we use in mathematics, but it was taught to my class as if it were a law. I am not going to say that everyone was taught this procedure the same exact way in America and that is our downfall. Nor am I trying to say that the teacher I had was a poor teacher. She was actually a very passionate teacher who invested a lot of time into teaching her students. It’s simply the fact that this is how she was taught how to multiply and therefore it is how she taught us. No one ever explained to her a better representation for this concept. The basic idea behind this procedure is that we are not multiplying $56 * 8$ but rather $56 * 80$. The “skip a space” procedure serves as a placeholder for the ten’s place. The actual process we are doing is:

$$56 * 84 = 56 * (4 + 80) = (56 * 4) + (56 * 80) = (56 * 4) + \big((56 * 8) * 10\big) = 224 + (448 * 10) = 4704$$

This is the sum of partial fractions represented below:



This is actually a bigger concept than it may seem, because with the right conceptual knowledge of arithmetic this problem (and even integers with greater number of digits) is solvable in your head or with little work needed. It's all about how you choose to distribute your numbers. Here are a few ways one could solve this problem with little work needed:

1) Multiplying by 2 is pretty easy with integers with enough practice for anyone. It's also common for people to realize that $4 = 2 * 2$ and $8 = 4 * 2 = 2 * 2 * 2$. Therefore, we have that
$$56 * 84 = 56 * (4 + 80) = (56 * 4) + (56 * 80)$$
Which is the same as before, just breaking the 84 into nicer "pieces". Observe:
$$(56 * 4) + (56 * 80) = \big((56 * 2) * 2\big) + \Big(\big((56 * 2) * 2\big) * 2\Big) * 10$$
As complicated as that may look, it's actually the same computation.
$$56 * 2 = 112 \;\rightarrow\; 112 * 2 = 224 \;\rightarrow\; 224 * 2 = 448$$
These are all the numbers you would need to solve this question:
$$\big((56 * 2) * 2\big) + \Big(\big((56 * 2) * 2\big) * 2\Big) * 10 = (112 * 2) + \big((112 * 2) * 2\big) * 10$$
$$= 224 + (448 * 10) = 224 + 4480 = 4704$$

There is another, easier way to solve this problem and it deals with breaking apart integers:

2) Let's use the commutative property and say $56 * 84 = 84 * 56$. Multiplying by tens is easy and multiplying by 5 is the same as multiplying by ten and then halving (This is not necessarily part of this problem; it is just the way I view multiplication of 5's.). Therefore,
$$84 * 56 = 84 * (50 + 4) = (84 * 50) + (84 * 6)$$
I know that $84 * 100 = 8400$ and therefore $84 * 50$ is half of 8400 which is 4200. If we know that $84 * 50 = 4200$ then $84 * 5 = 420$. So,
$$84 * 6 = 84 * (5 + 1) = (84 * 5) + 84 = 420 + 84 = 504.$$
Thus we have
$$84 * 56 = (84 * 50) + (84 * 5) + (84 * 1) = (84 * 5) * 10 + (84 * 5) + (84 * 1)$$
$$= 4200 + 420 + 84 = 4200 + 504 = 4704$$
We got this answer only by working with 5's and 10's.

So far when we have represented the multiplication of the 7's table as well as doing mental arithmetic with multi-digit numbers, we have used a property that we have yet to state. When we discussed $7 * 8$, I showed how we can view this equation as $(5 * 8) + (2 * 8)$. On these last two examples of how to multiply, we also used the same idea to "break up" numbers in ways that made it easier to multiply. This idea is known as the Distributive property and it is one of the most essential

concepts for the multiplication of integers. The basic definition of the Distributive property is that if you have any three numbers A, B, and C then $A * (B + C) = AB + AC$.

The most important aspect for this property I believe is that it allows the ability to alter numbers creatively to produce answers that would otherwise be very difficult or impossible. We have shown situations where we this is true. I was never shown this use for the Distributive property in school. I was simply told what its definition was and then given numbers to apply it with. The direct application of this property where you are given numbers A, B, and C and asked to show the truth of this property isn't as conceptual as the reverse where you are given two integers D and E and asked to solve mentally. $7 * 8$ is a basic start but there are greater values that are very basic to solve using this property. For example, say you don't know the answer to 12^2 and you are asked to solve this problem mentally. Well,

$$12^2 = 12 * 12 = 12 * (10 + 2) = (12 * 10) + (12 * 2) = 120 + 24 = 144.$$

That is a solution that I believe most people could arrive at without paper or a calculator with the right teaching. Here are a few more examples of how the distributive property can benefit arithmetic.

1. $17 * 9 = 17 * (10 - 1) = (17 * 10) - (17 * 1) = 170 - 17 = 153$
2. $101^2 = 101 * 101 = (101 * 100) + (101 * 1) = 10100 + 101 = 10{,}201$
3. $18 * 22 = (18 * 20) + (18 * 2) = (18 * 2 * 10) + 36 = 360 + 36 = 396$
4. Tipping 15% at a restaurant:
$$(Total\ Bill) * 15\% = (Total\ Bill) * 0.15 = (Total\ Bill) * (.10 + .05)$$
$$= (Total\ Bill) * .10 + (Total\ Bill) * .05$$
Therefore, say your check was \$32.40:
$$32.40 * 0.15 = (32.40 * 0.10) + \frac{32.40 * 0.10}{2} = 3.20 + \frac{3.20}{2} = 3.20 + 1.60 = \$4.80$$
Another way to compute tips is getting 10% and then doubling your answer to get 20%, then estimating the median in between the two answers:
$$32.40 * 0.10 = 3.20$$
$$3.25 * 2 = 7.40$$
$$32.40 * 15\% \approx \$5.00$$

Also, with additional help from the Associative property, we can solve even more multi-digit computation easily. Consider the expression $250 * 4$ and $36 * 25$. We will show two ways for solving both equations:

1. $250 * 24 = 250 * (4 * 6) = (250 * 4) * 6 = 1000 * 6 = 6{,}000$
2. $250 * 24 = \left(1000 * \frac{1}{4}\right)(4 * 6) = 1000 * \left(\frac{1}{4} * 4\right) * 6 = 1000 * 6 = 6{,}000$
3. $36 * 25 = (9 * 4)(5 * 5) = 9 * (4 * 5) * 5 = 9 * (20 * 5) = 9 * 100 = 900$
4. $36 * 25 = 36 * \left(100 * \frac{1}{4}\right) = (36 * 100) * \frac{1}{4} = 3600 * \left(\frac{1}{2} * \frac{1}{2}\right) = \left(3600 * \frac{1}{2}\right) * \frac{1}{2} = 1800 * \frac{1}{2} = 900$

These four solutions are much easier to solve and they are derived from a fundamental, conceptual understanding of how the Associative and Distributive property truly operate. To understand mathematics conceptually means that you understand the fashion in which mathematics behaves. A lot of the times and mathematical problem can be altered in appearance by some clever application of a property or procedure to arrive at an answer faster or even at all. However, if a person does not truly comprehend how that property or procedure operates then they will probably have a tough time noticing when it is beneficial to apply. Therefore the person has a tool in their toolbox with no idea how to use it. That defeats the purpose of having that tool in the first place. There is no point of having to memorize a procedure or formula if a student is not given the oppurtunity to explore and discover its uses and applications. Thus, a student needs a conceptual understanding of ideas in mathematics so they can learn to recognize when they are needed and/or useful.

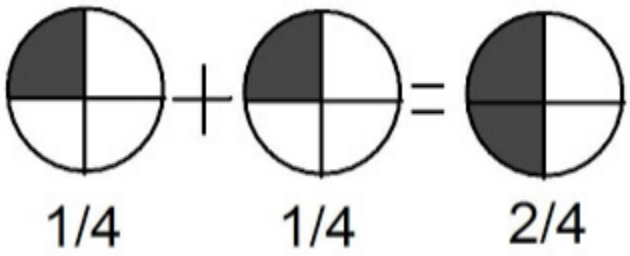
How Procedural Knowledge Caues Misconceptions with Fractions

Students, as well as teachers, in America have shown to have major misconceptions dealing with operations on fractions. While teachers know how to add and subtract fractions, there are a few misconceptions that students make (Association of Middle Level Education, February). Major issues arise when teachers and students are asked to operate with fractions. First, I will start with the misconceptions students can make when addition fractions.

The cardinal rule that I was given when I was being taught to add fractions was that, "the only way to add fractions is if they have the same denominator; then you add the numerators." From my own experience, I had no real knowledge of which number was the numerator and which was the denominator besides one number was on top of the fraction and the other was on the bottom. That explanation was just words on a page and did not leave me with any understanding of what it meant to add fractions. If I were shown visually how to add fractions, the terms "numerator" and "denominator" would have been irrelevant to my understanding. However, the basic idea is:

If you have three numbers A, B, and C, then $\frac{A}{B} + \frac{C}{B} = \frac{A+C}{B}$.

This property isn't too hard to comprehend and it can be shown visually like so:



This is just an example of $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, however this idea can be created with other numbers and shown in the same fashion to build an understanding. There are also better methods to explain adding fractions.

I'm going to build the concept of adding fractions using a model shown to me by an article written by Randolf A. Philip. In his article, Dr. Philip uses a hexagon to represent a whole unit (or the number 1), and then he uses smaller shapes to represent his fractions. Here's his different shapes and their respective quantities:

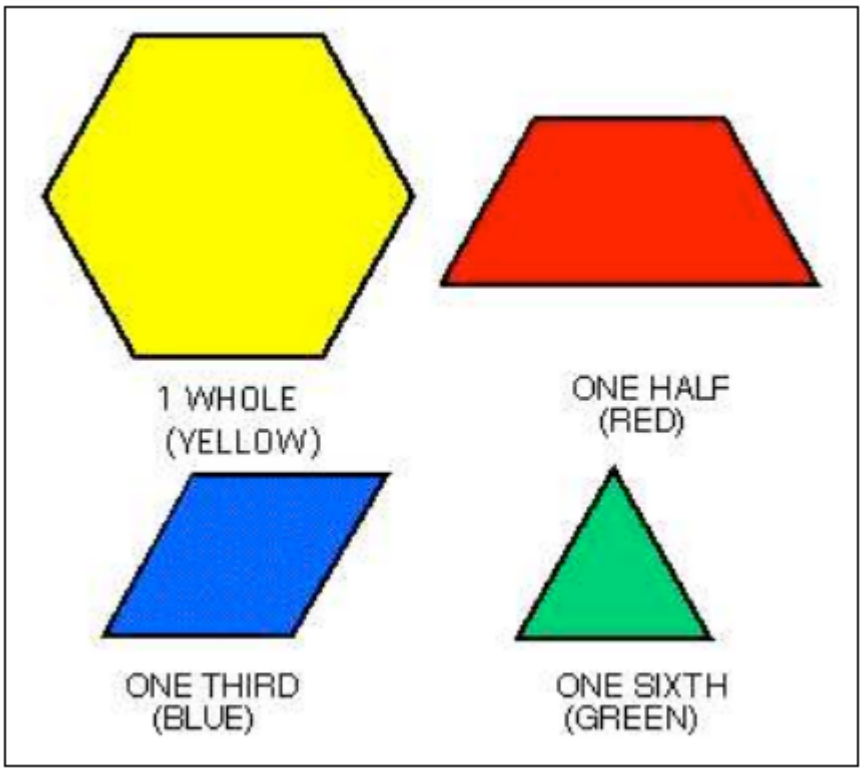


Figure : (Philip, 2000)

Dr. Philip first asked students to find different combinations to make up one whole hexagon. Then, he asked for the students to continue to find connections with other pieces; i.e. find what combinations can make up a trapezoid or a rhombus. Therefore, Dr. Philip had his students adding and subtracting fractions before they were ever presented the concept directly. Students started to make some connections like these:

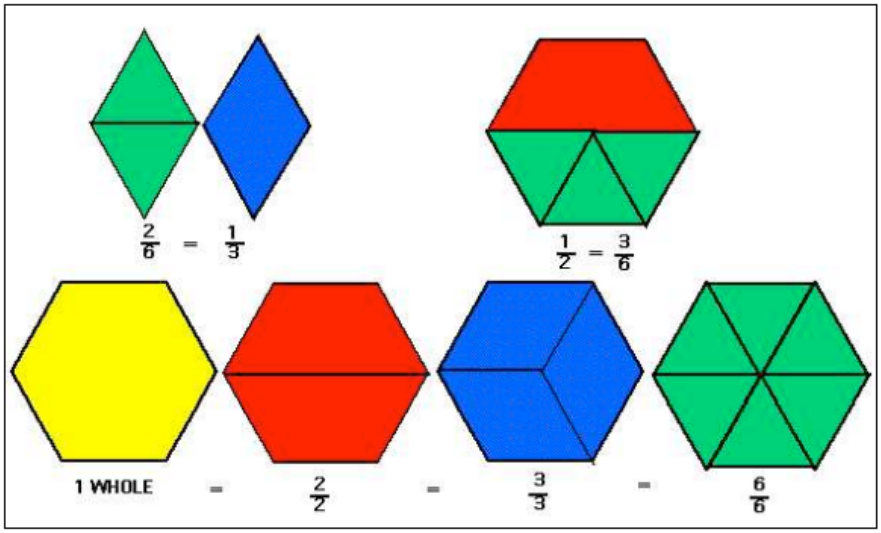


Figure: (Philip, 2000)

When the students became comfortable with dealing with these different shapes and their representations relative to the whole, Dr. Philip gave the students the problem $\frac{1}{2} + \frac{1}{3}$ to solve. Here's how two students solved for this equation:

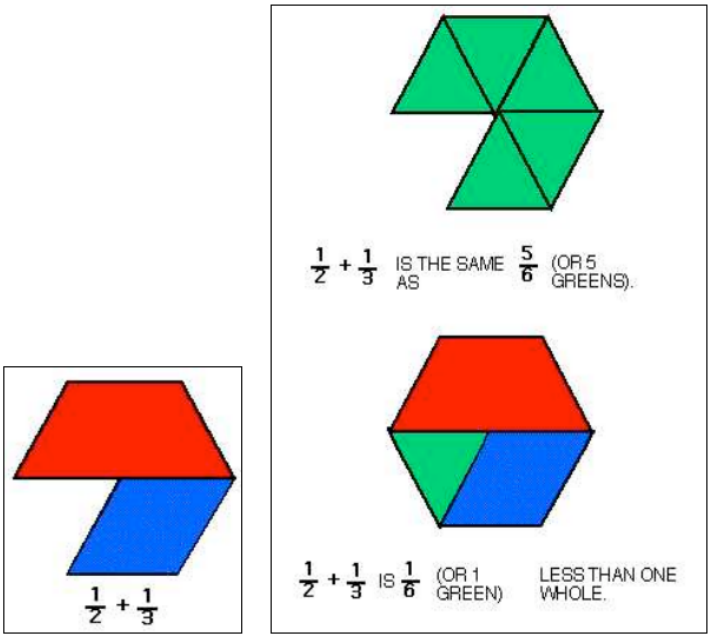


Figure: (Philip, 2000)

The beauty about this presentation to adding fractions is that it incorporates all of the concepts needed to complete the procedure with a rationale as to why it works. The first student understood that $\frac{1}{2} = \frac{3}{6}$ and that $\frac{1}{3} = \frac{2}{6}$. Therefore, this student actually computed for the least common multiple (LCM) and this used the procedure I stated earlier and arrived at their answer of $\frac{5}{6}$. This is the common way to add fractions procedurally, but being able to see this process visually by a student's own personal perspective increases their chances to either memorize the concept or be able to draw back on this lesson and recreate their understanding. The second student decided to look at what $\frac{1}{2} + \frac{1}{3}$ was relative to the whole and noticed that $\frac{1}{2} + \frac{1}{3} = 1 - \frac{1}{6}$. This is a process that different than my thinking style but it is something that this student could learn to use in certain situations. Here's one way to arrive at that solution computationally:

$$\frac{1}{2} + \frac{1}{3} = 0 + \left(\frac{1}{2} + \frac{1}{3}\right) = \left(\frac{1}{2} - \frac{1}{2}\right) + \left(\frac{1}{2} + \frac{1}{3}\right) = \left(\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{2} + \frac{1}{3}\right) = 1 + \left(-\frac{3}{6} + \frac{2}{6}\right) = 1 - \frac{1}{6} = \frac{5}{6}$$

This process requires adding a zero to the equation in the form of $\frac{1}{2} - \frac{1}{2}$. This gives you a whole subtracted by a fraction which is not a very difficult equation to compute. This process still requires the use of the LCM so it might be simpler to arrive at the solution in the fashion as we did before (student 1). However, if this is how a student sees and understands the addition of fractions in any given scenario then it's just as effective as any other method at giving the student the correct solution. There is a fair possibility that the equation I used to compute this method of thought is not the most effective procedure for showing $\frac{1}{2} + \frac{1}{3} = 1 - \frac{1}{6}$, or that there is a possibility that a student simply visualizes this concept more effectively this way. The one thing that is for certain though is that these students are building a strong rationale for how the addition of fractions procedure works.

When addition of fractions is taught from only a procedural perspective, a couple problems tend to occur with students when it comes to adding fractions together. For one, students tend to get this concept confused with the multiplication of fractions Therefore if we have four numbers A,B,C, and D, students make the error

$$\frac{A}{B}+\frac{C}{D}=\frac{A+C}{B+D}$$

For a simple counter-example, if we have $\frac{A}{B}=\frac{1}{2}$ & $\frac{C}{D}=\frac{1}{2}$, then $\frac{1}{2}+\frac{1}{2}=\frac{2}{4}=\frac{1}{2}$ with this misconception. This is why the visual representation benefits students because if shows them the combination of fractions and how they relate to one another. If you can show students that three triangles (which is represented as $\frac{3}{6}$) equals one trapezoid ($\frac{1}{2}$) then hopefully minor mistakes such as adding the denominators are made less often. The reason being is that students will be able to visualize $\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{1}{2}$, and therefore not be led to believe it equals $\frac{3}{18}$ (which note that $\frac{3}{18}=\frac{1}{6}$) . Another common mistake is the reverse of the rule; that is where students have similar denominators and therefore add the denominators together:

$$\frac{A}{B}+\frac{A}{C}=\frac{A}{B+C}$$

If both integers are positive, then this actually always produces an answer less than both of the two fractions. However, the reason this mistake is made is because it looks so much like the actual property. It's a simple mistake that can easily be disregarded with better representation. You wouldn't say a blue plus a blue ($\frac{1}{3}+\frac{1}{3}$) is equivalent to a green ($\frac{1}{6}$), because one blue is already bigger than one green. However, when students are just given the property and asked to memorize the formula, this mistake easily comes into the realm of possibility. An example of how misconstrued this concepts can become comes for a journal article where a study was done on twelve year old boy by the name of Benny how

was seemingly excelling in his math classes. However, Benny was overwhelmed with so many different procedures and formulas for mathematics that he had completely convinced himself that math behaved in a very abstract way. Many of Benny's formulas were correct in using for some mathematical procedure, but Benny had come to believe that many of these procedures worked in incorrect situations. Here's a small excert from the article:

~~example, $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$~~ Benny believed that there were rules for different types of fractions, as illustrated by the following excerpt:

B: In fractions we have 100 different kinds of rules....

E: Would you be able to say the 100 rules?

B: Ya ... maybe, but not all of them.

He was able to state addition rules for fractions clearly enough for me to judge that they depended on the denominators of the fractions and were equivalent to the following:

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}, \quad e.g., \quad \frac{3}{10}+\frac{4}{10}=\frac{7}{10};$$
$$\frac{a}{b}+\frac{c}{d}=\frac{a+c}{b+d}, \quad e.g., \quad \frac{4}{3}+\frac{3}{4}=1;$$
$$\frac{a}{b}+\frac{c}{c}=1\frac{a}{b}, \quad e.g., \quad \frac{2}{3}+\frac{4}{4}=1\frac{2}{3};$$
$$\frac{a}{10}+\frac{b}{100}=\frac{a+b}{110}, \quad e.g., \quad \frac{6}{10}+\frac{20}{100}=\frac{26}{110}.$$

Figure: (Erlwanger, 1973)

Note that Benny was a student who was considered to be excelling in mathematics. Meanwhile, Benny was creating his own rationale for concepts by combining different procedures and formulas together. There were situations where Benny would get multiple answers for one problem dealing with addition of fractions and when asked how one equation could equal multiple things, Benny should argued that the answers "were the same! But different."

The biggest concern with teaching the operation of fractions is division. Teachers themselves have a high level of misconceptions and they teach it the same way they learned it. Teaching division of fractions often does not go further than the sayings, “Invert and multiply,” or “Keep it, change it, flip it.” This idea lacks any depth needed to use this concept in any application. Out of twenty-three American teachers surveyed by Ma (1999), only one teacher could create a word problem that required the use of division of fractions. Out of that same twenty-three, only five teachers could give any justifiable reason to why the “invert and multiply” algorithm worked and none of these teachers had any other approach to divide fractions. Conceptual understanding has to begin with the teacher before it can be conveyed to the students. The “invert and multiply” method is a very useful tool for division, but it creates no understanding for how fractions actually behave with division. Therefore it is imperative that teachers have the capability to convey ideas in multiple perspectives and present these approaches to students before they give students procedures. A lot of the times students can derive this procedure independently of being told the method. Going back to the methods of Dr. Philip, he asked his students a word problem that asked students if you had $1\frac{1}{2}$ cups of sugar with a recipe calling for $\frac{1}{3}$ cups of sugar per recipe, how many recipes can you make? Here’s his visual representation:

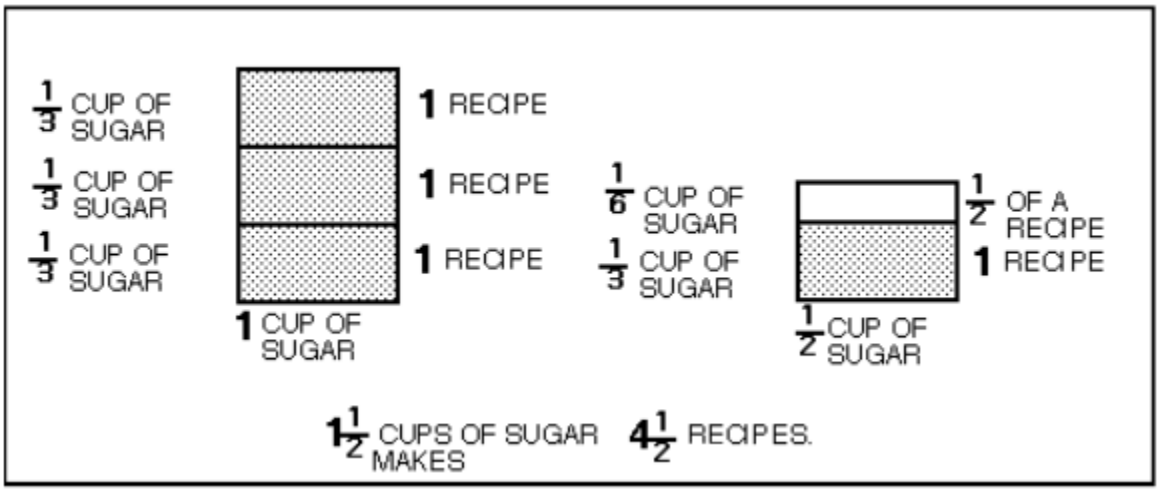


Figure: (Philip, 2000)

Dr. Philip already knew that his students understood that $\frac{3}{3} = 1$ and that $\frac{1}{3} < \frac{1}{2}$ due to their work with the shapes. Therefore, the students argued that they can definitely make at least 4 recipes with $\frac{1}{6}$ left over. The students also knew that $\frac{1}{6}$ was halve of $\frac{1}{3}$. Therefore, the students make the conclusion that the answer was $4\frac{1}{2}$. These students were dividing fractions without even realizing it. The procedure "invert and multiply" has no meaning when it stands alone as to how to divide fractions. The procedure is nothing more than an aid for the concept that Dr. Philip showed his students. Also, the procedure doesn't help a student understand when it is necessary to divide or multiply fractions in a word problem because the two procedures can seem very similar in their wording. On a personal note, I did not fully understand division of fractions (nor did I realize I didn't) until I saw Dr. Philips representation. I will show you one more picture from Dr. Philips that fully developed my conceptual understanding of division of fractions:

Use the picture at the left to explain why $1 \div 4/5 = 5/4$.

It's easy to see that $1 \div \frac{4}{5}$ makes at least one recipe. After that, you would have $\frac{1}{5}$ left over for another recipe. So, you have $\frac{1}{5}$ cups of sugar and you need $\frac{4}{5}$. Therefore, you have $\frac{1}{4}$ of another recipe. Thus, $1 \div \frac{4}{5} = \frac{5}{4}$.

It is the pushing for the rationalization of ideas like these that makes mathematics understandable and enjoyable. Simply giving students formulas and procedures in not only a redundant process of memorization, but it limits the possibilities that the concepts are applicable outside of a

direct algorithm equation. Mathematics is far too interesting of a subject to be treated as though it only consists of procedures. Mathematics has a behavior that is very interesting to learn and discover. However, it is all about how teachers choose to present these ideas to their students.

The Time For a Change

If there ever was a time to make a commitment to changing how we teach mathematics it is now. Teachers haven't failed at teaching mathematics in America. They have always taught to standards and mostly have taught them well. The problem was how standards demanded mathematics to be taught. The standards had been forcing teachers to cover such a broad array of concepts in a given year that it seemed like teachers could not afford the time or effort to focus on conceptual understanding. Now, with the change in standards across America, standards are slowly evolving into a more conceptual approach to teaching mathematics. In Georgia, standards have changed to allow teachers to teach fewer concepts. This creates the time and capability to teach the fewer concepts more conceptually. Standards are also asking that teachers do present concepts in multiple approaches. This is a significant change and a start of a new era in teaching that hasn't changed much in its lifespan. However, this is not going to be an instantaneous overhaul to teaching. To change a behavior takes time and this is one that every America has grown accustomed to with mathematics. The last standards Georgia was given didn't even get the timeframe to show if it was effective. Those standards came and went in three years, not allowing to a graduating class make it through the full four years of high school. It is irrational to believe that there is an instant cure to the lack of understanding in mathematics. Teaching conceptually is changing the culture of how people view what it means to do math. Therefore these news standards, called the Common Core, needs the time necessary to make this change and the commitment to see it implemented.

Common Core has now been adopted by 45 states in America and D.C. There are 6 conceptual categories that CC has:

- Number and Quality
- Algebra
- Functions

- Modeling
- Geometry
- Statistics and Probability

With these 6 categories is coming a new mindset that students should be focused on developing a rationale for the concepts they are learning. However, now I believe comes a very critical moment in our education of mathematics. Will these new standards be given the time necessary to make a true change in the way in which mathematics is taught in America? The last set of standards was not even allowed a complete high school graduating class before it was thrown out of the school system. An overhaul to teaching mathematics cannot happen over night. This is a public opinion and attitude developed by the Americans for generations that mathematics is centered around procedures and formulas. Some students who are already in the school systems will pick up these new ways of understanding mathematics and will excel at it. However, there will be other students who already lack a foundational understanding of mathematics from previous years in the school system that will continue to struggle and may even struggle more. This new way of teaching has to reach students in the earliest grades possible to have a greater effect on high school students. Students and parents should not be too quick to write off this new approach because it is abstract to how they learned mathematics previously. Mathematics is mentally stimulating and enjoyable to understand and should be given the respect and attention it deserves to be presented to students in its proper fashion. These new standards may not be perfect in every aspect, but they are a start to a new way of presenting mathematics. If America can learn to change its mindset on mathematics and what it means to do math, then America can be on a track destined to create great future mathematicians; with great mathematicians comes the possibities to innvovate and create new technologies and advancements in so many different realms of thought. It is time to make a commitment to change and allow teachers the time and resources to better themselves as well as their students in mathematics.

Concluding Statements

My argument is that for generations now, America has been taught to believe a major misconception about mathematics. Through no fault of the teachers, we have been led to believe that mathematics lacks mental stimulation outside of memorization. As I said before, mathematics is actually at its truest essence about creation and exploration of ideas and concepts. While I do agree that mathematics can be frustrating and sometimes cruel, I believe that for reasons that differ from many Americans. Mathematics frustrates me to answer its riddles and puzzles that seem to happen with any mathematical concept. I have a desire to understand and rationalize mathematics, and while I'm not in any way demanding that everyone love math the way I do, I want to see at the very least students be given the oppurtunity to see math this way. There's a doorway to a beginning for Americans to start being given the oppurtunity to conceptually understand mathematics, and I believe that right now we are on the threshold. China has created a system that pushed teachers as well as students, and I believe we should incorporate the same strategies in our school systems as well. If we continue to challenge students to comprehend mathematics, that not only opens more possbilities for the students to explore a higher education, but it also creates oppurtunities for America to advance as a nation. There is a reason mathematics is called the language of nature. Through mathematics anything can be explained. Who knows what inventions or developments future mathematicians can create, but the oppurtunity to create those mathematicians has to be in place for generations to come.

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