Understandings and Misunderstandings of Trigonometry

Christopher Williams

Georgia College & State University

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Abstract

Trigonometry plays a major role in our society. Trigonometry is the study of triangles and the relationship between the measures of its angles and sides. In this research, we will provide information concerning the misconceptions of basic concepts in trigonometry. The study will also identify whether students have a conceptual understanding of those concepts or just a surface level understanding. The study will provide information concerning the misconceptions of trigonometry in math courses that could help fellow educators.

Understanding and Misunderstandings of Trigonometry

Introduction

Trigonometry plays a major role in our society. Weber states in an article, "trigonometry is one of the earliest branches of mathematics topics that links algebraic, geometric, and graphical reasoning, it can serve as an important precursor towards understanding pre-calculus and calculus" (Weber, 2005, p.91). Trigonometry is the study of triangles and the relationship between the measures of its angles and sides. I was first introduced to trigonometry in my 8th grade year of middle school. I became interested in learning trigonometry, considering that I had a difficult time understanding how measurements of a triangle can be related and how they correspond to the unit circle. After many tries and attempts, I finally understood the concept. My motive for doing this study comes from personal experiences which I have previously described. I feel that we need this study considering that, without an understanding of the unit circle, trigonometry functions, and relations among triangles, many architects, draftsmen, engineers, pilots, game developers, and even chemists would not be able to complete task that involve trigonometry. This study will provide information concerning the misconceptions of basic concepts in trigonometry. The study will also identify whether students have a conceptual understanding of those concepts or just a surface level understanding. The study will provide information concerning the misconceptions of trigonometry in math courses that could help fellow educators. The research questions that I have investigated are as follows:

- Do students who have taken a trigonometry course have a conceptual understanding of trigonometric functions?

- Do students who have taken a trigonometry course have a conceptual understanding of radian angle measures?
- What misconceptions do student have about the basic trigonometric ratios and radian angle measures?

Literature Review

Procedural Fluency and Conceptual Understanding

An article states that “no term captures completely all aspects of expertise, competence, knowledge, and facility in mathematics, we have chosen mathematical proficiency to capture what we believe is necessary for anyone to learn mathematics successfully”(Kilpatrick, Swafford, & Findell ,2001, p.116). Mathematical proficiency consists of five strands of learning which are: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. I will be focusing on two concepts of mathematical proficiency which are: conceptual understanding and procedural fluency. Conceptual understanding can be described as functional grasp of mathematical ideas” (Kilpatrick, Swafford, & Findell, 2001, p.118). In other words, to have conceptual understanding, an individual has skillful knowledge of mathematical concepts in the way that they can apply concepts to something more than just surface level. Also, individuals that have conceptual understanding, can use the ideas and concepts that they already know to learn new ideas” (Kilpatrick, Swafford, & Findell, 2001, p.118). A good indicator that an individual has conceptual understanding is when the individual can manipulate various mathematical concepts, while knowing how manipulating these concepts can be useful for different purposes. “To find one’s way around the mathematical terrain, it is important to see how the various representations connect with each other, how they are similar, and how they are different. The degree of students’ conceptual understanding is related to the richness and extent of the connections they have made" "(Kilpatrick, Swafford, & Findell, 2001, p.118). For example, a student may be asked to find the sin30º, the student convert degrees to radians. By the student converting to radians, he or she may make the connection that $\frac{\pi}{6}$ is on the unit circle and to find the sin of $\frac{\pi}{6}$ all they would have to do is look at the y-value of $\frac{\pi}{6}$ on the unit circle to solve the problem. An accurate understanding of knowledge that has been learned, can help aid in providing a basis for understanding unfamiliar problems and knowledge. By having a conceptual understanding of material can result in one not having to learn as much, considering that they can identify deeper similarities among unrelated situations. One article states, "There is a broad consensus among mathematics education researchers that the goal of mathematics courses is not only for students to memorize procedures and acquire reliable methods for producing correct solutions on paper and pencil exercises, rather students should learn mathematics with understanding" (Weber, 2005, 92).

Procedural fluency can be describe as having the knowledge of procedures. In other words, meaning that one can use procedures appropriately, accurately, and is skill at performing them efficiently" (Kilpatrick, Swafford, & Findell, 2001, p.121). There are many tasks that involves mathematics in everyday life which require facility with algorithms for performing computations either mentally or in writing" (Kilpatrick, Swafford, & Findell, 2001, p.121). "Some algorithms are important as concepts in their own right, which again illustrates the link between conceptual understanding and procedural fluency" (Kilpatrick, Swafford, & Findell, 2001, p.121). If one knows how to do a procedure without an understanding, it can lead to it being difficult for one to understand the reason behind the procedure. Without a good grasp of procedural fluency, one may have a hard time having an enough understanding of ideas. In other words, if a student can recall a math result by using a procedure, it may prevent them from seeing relationships among the result. "When skills are learned without understanding, they are learned as isolated bits of knowledge. Learning new topics then becomes harder since there is no network of previously learned concepts and skills to link a new topic to. This practice leads to a compartmentalization of procedures that can become quite extreme, so that students believe that even slightly different problems require different procedures" (Kilpatrick, Swafford, & Findell, 2001, p.123). Also, one who has learned a procedure without an understand, often cannot do any more than apply the procedure, compared to someone who understands a procedure and can manipulate a procedure to make it easier to use.

Developing Mathematical Proficiency

"Proficiency in teaching is related to effectiveness: consistently helping students learn worthwhile mathematical content. Proficiency also entails versatility: being able to work effectively with a wide variety of students in different environments and across a range of mathematical content" (Kilpatrick, Swafford, & Findell,2001, p.369). It is significant for teachers to help their student to develop and become proficient in math, considering that it will help students develop a conceptual understanding of the material learn, also they will be able see relations among concepts especially in trigonometry. There are three areas of knowledge that are crucial for teachers teaching mathematics, they are knowledge of mathematics, knowledge of instructional practices, and knowledge of students.

Teachers need knowledge of mathematics, considering that it includes mathematical facts, concepts, procedures, and relationships. If teachers do not have a good foundation in the knowledge of mathematics, then teachers can not help their students develop mathematical proficiency. In an article it states that "teachers may know the facts and procedures that they teach but often have a relatively weak understanding of the conceptual basis for that knowledge" (Kilpatrick, Swafford, & Findell, 2001, p.372).

Teachers knowledge of mathematics is directly tied to their instructional practice. If teachers only have a surface level understanding of a mathematical concept, it will be hard teachers to provide an explanation for that concept, and actively engage their students to see other way of looking at the concept. "Researchers found that teachers with a relatively weak conceptual knowledge of mathematics tended to demonstrate a procedure and then give students opportunities to practice it. Not surprisingly, these teachers gave the students little assistance in developing an understanding of what they were doing. When the teachers did try to provide a clear explanation and justification, they were not able to do so. In some cases, their inadequate conceptual knowledge resulted in their presenting incorrect procedures" (Kilpatrick, Swafford, & Findell, 2001, p.378).

The most important area that teachers need in order to help their students develop mathematical proficiency is having a knowledge of students. "The teacher needs to know something of each student's personal and educational background, especially the mathematical skills, abilities, and dispositions that the student brings to the lesson" (Kilpatrick, Swafford, & Findell, 2001, p.378). This means that the teacher knows their students and how students tend to learn in general. Teachers will have some idea of what students may have misunderstanding with and find ways that will be more efficient of teaching a math concept to their students.

Misconceptions

As a math student, a student's thinking consists of formulas, and relevance. A problem that sometimes makes learning mathematics difficult is students' misconceptions from previous inadequate teaching, informal thinking, or poor remembrance. A misconception is a mistaken idea, from a misunderstanding of something (Orhun, p.208). In other words, one may have a conceptual misunderstanding about a how to solve a math problem. When it comes to learning trigonometry, often misconceptions arise from the way the teacher teaches. “The problem is how these topics could be taught i.e., how they could be presented in classroom. The impression is that trigonometry is generally taught via teacher-active method and memorizing the ready knowledge and repeating them, the students learn trigonometry. It is known that this learning is generally active in brief term and it is hard to transfer the principle learned to new situations” (Orhun, p.210). In other words, students are developing a procedural understanding, and can only apply their memorize knowledge to concepts that they are familiar with. An author states that we can eliminate some of the misconceptions by teaching trigonometry definitions first and then applying them to angles, some of the misunderstanding would not be there (Orhun, p.211).

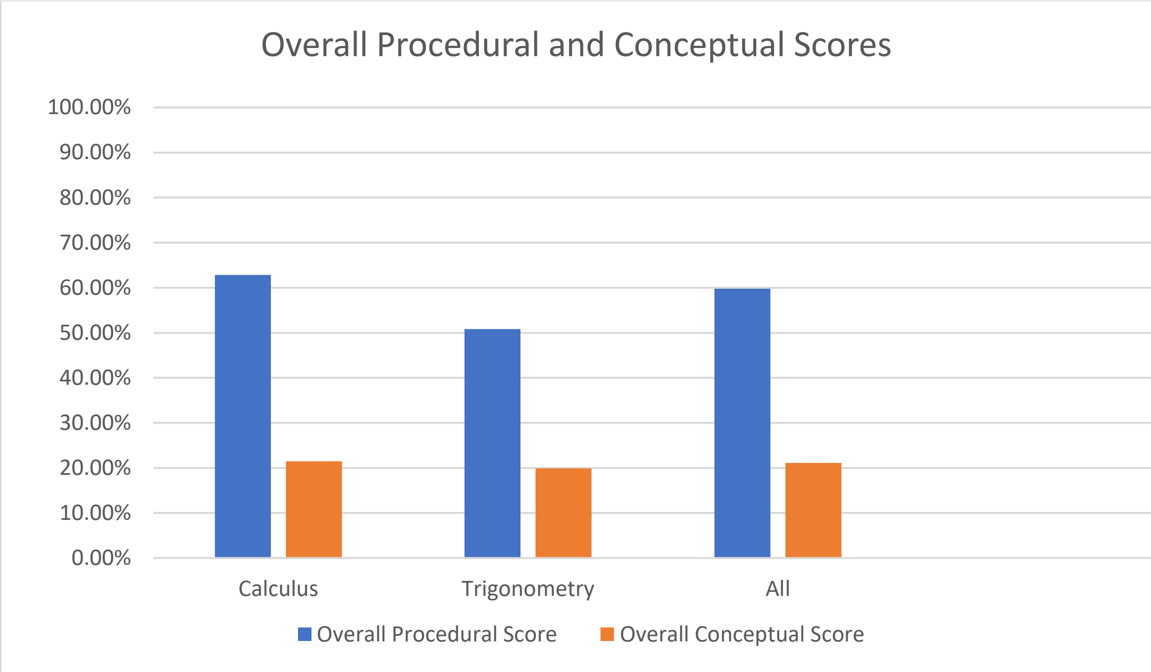
Methods

My research was conducted in Math 1112 “Trigonometry”, and Math 1261 “Calculus”. The participants in my study were voluntary and an explanation of the researched that was conducted prior to participants receiving a consent form. The method that I used to gather research is by conducting an assessment call Trigonometry Assessment, which allowed me to assess participants conceptual understanding of trigonometric functions/ angle measure, as well as the methods individuals used to solve various trigonometry problems. The assessment consisted of twelve questions. Questions 1-8 are focused on trigonometric functions. For example, the first question on the assessment focused on whether students had procedural fluency for what each of these functions are equal to or they did not. The second question on the assessment showed if individuals had a conceptual understanding on how to estimate trigonometry functions and give an explanation on arriving at their solution. Questions 9-12 are focused on angles. The way I analyze the data was by constructing a rubric which is broken down into two sub rubrics procedural fluency and conceptual understanding rubrics. Each question a student could earn a total of two points. To get the full two points students had to give a correct or reasonable answer that also include a good justification for their answer. Students received a point if they could answer the question correctly without giving any justification. Students received a half point if their work or justification was okay but ended up not getting the correct answer. Students received no points if they did not answer the question correctly and the justification was not reasonable. For questions 1, 4, 6, 10 I use procedural rubric to grade these; questions 2, 3, 5, 7, 8,9,11, and 12, I use the conceptual rubric to grade these. There was a total of 60 students that I gave the assessment to, 44 of the 60 were calculus students and 16 were trigonometry students. Students were given one class period to answer the questions on the assessment.

Findings

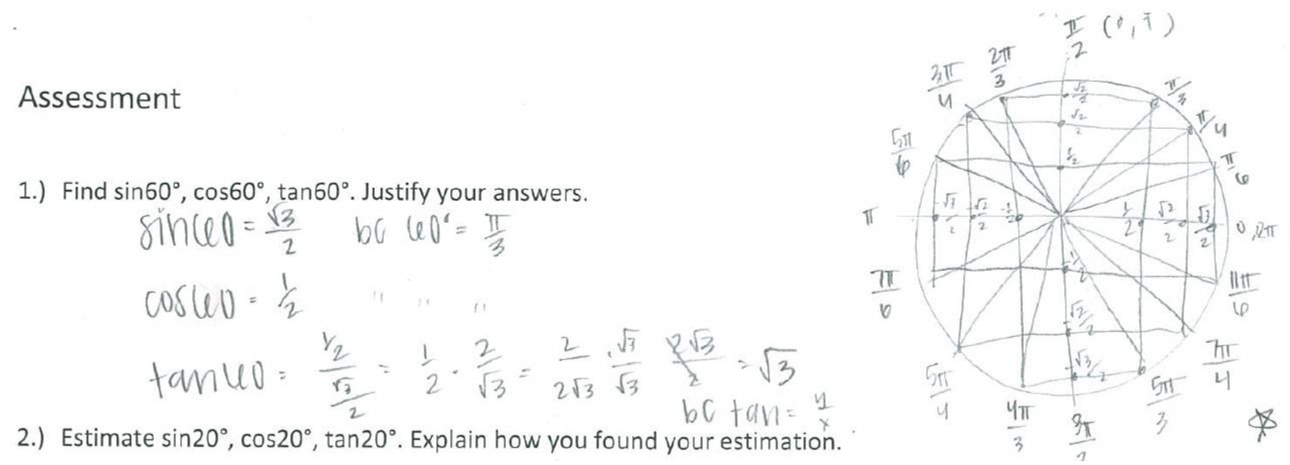
Conceptual Understanding

The assessment gave me the opportunity to see whether students had a conceptual understanding of trigonometry or were they able to follow a procedure. As stated earlier, I wrote the assessment in a way such that some questions would show if students had procedural fluency of a concept of trigonometry, and other questions checked to see if the students had a conceptual understanding of the concept. On the questions that followed a procedure, students tended to do better on them, than the ones that were conceptual. The graph below shows the overall scores for the procedural questions compared to the overall score for the conceptual questions.



By the graph, one can see that the overall procedural scores were nearly 3 times higher than the overall conceptual scores.

Now, we will discuss a few problems on the assessment. Question one asked students to find the sine, cosine, and tangent of a well-known angle which was 60 degrees. Question two asked students to estimate the sine, cosine and tangent of the angle 20 degrees. The goal of this question was to see if students could expand their knowledge of the trigonometric functions to angles that are not well-known. Many students could find the ratios for the 60 degree angle, but when they had to estimate for 20 degrees, many students had a hard time doing so. Below is typical example of a student's response to #1.


Assessment

1.) Find sin60°, cos60°, tan60°. Justify your answers.

$\sin 60 = \frac{\sqrt{3}}{2}$ bc $60° = \frac{\pi}{3}$

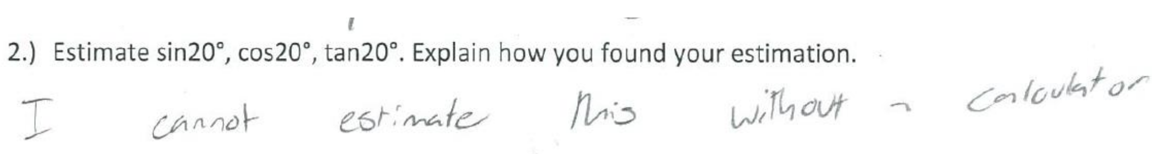
$\cos 60 = \frac{1}{2}$

$\tan 60 = \frac{1/2}{\sqrt{3}/2} = \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{2}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \; \frac{2\sqrt{3}}{2} = \sqrt{3}$

bc $\tan = \frac{y}{x}$

2.) Estimate sin20°, cos20°, tan20°. Explain how you found your estimation.

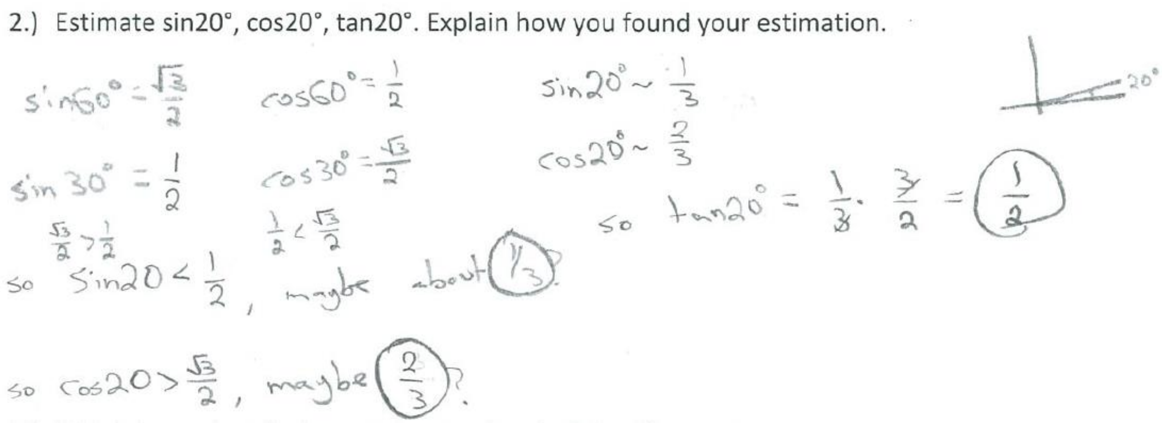
The student scored two points for their response. She showed that she can follow the procedure of recalling the memorized the unit circle and using basic algebra to solve for the trigonometric functions. A typical example of student work from #2 is below.


2.) Estimate sin20°, cos20°, tan20°. Explain how you found your estimation.

I cannot estimate this without a calculator

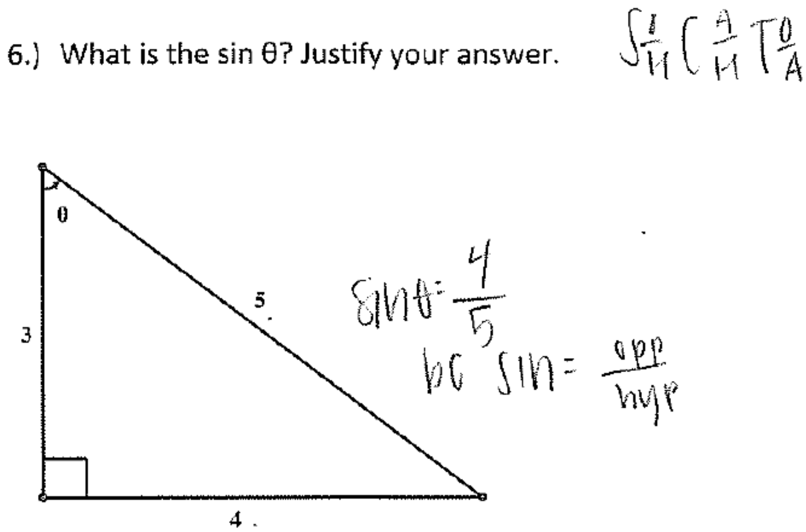
The student scored zero points for their response. The student could not use the memorized unit circle to approximate the trigonometric functions. This shows a lack of conceptual understanding, considering that if "knowledge that has been learned with understanding provides the basis for generating new knowledge and for solving new and unfamiliar problems" (Kilpatrick, Swafford, & Findell, 2001, p.119).

For question one, of the calculus students 43.6 % could answer this question and only 6.3% were able to answer question two. 43.8% of the trigonometry were able to give an answer to question one and only 8.3% were able to answer question two. The overall score for question one was 43.6% and the overall score for question two was 6.7%. Most students were able to do question one, considering that question one showed if students could follow a procedure by finding the measurements using an angle that is used a lot in trigonometry. In other words, students could use a procedure to find the sine, cosine, and tangent of common angles, but they can't expand their understanding to non-typical angles. This shows a lack of understanding of the meaning of the trigonometric functions, considering most common angles are clearly seen on the unit circle and students could draw the unit circle to find the answer to the trigonometric functions. The student work below is an example that shows a conceptual understanding of the trigonometric functions.

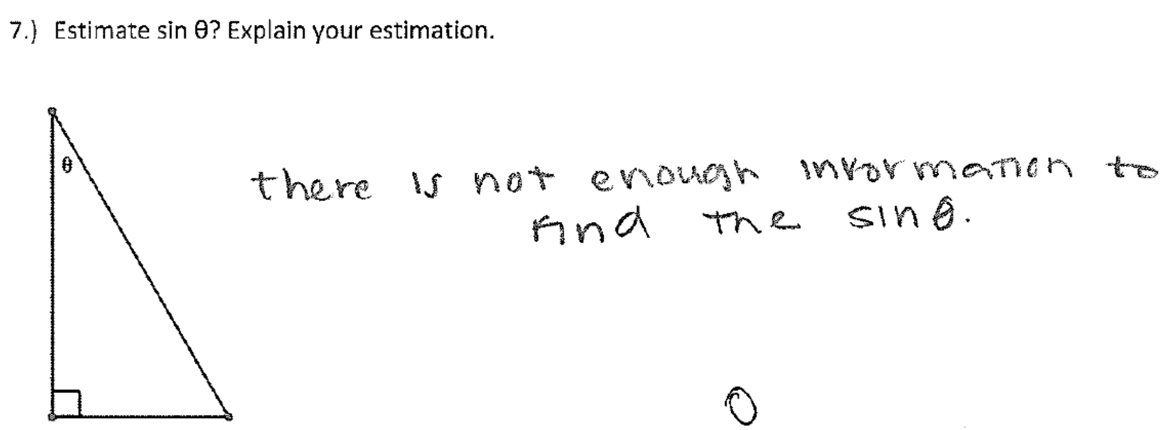


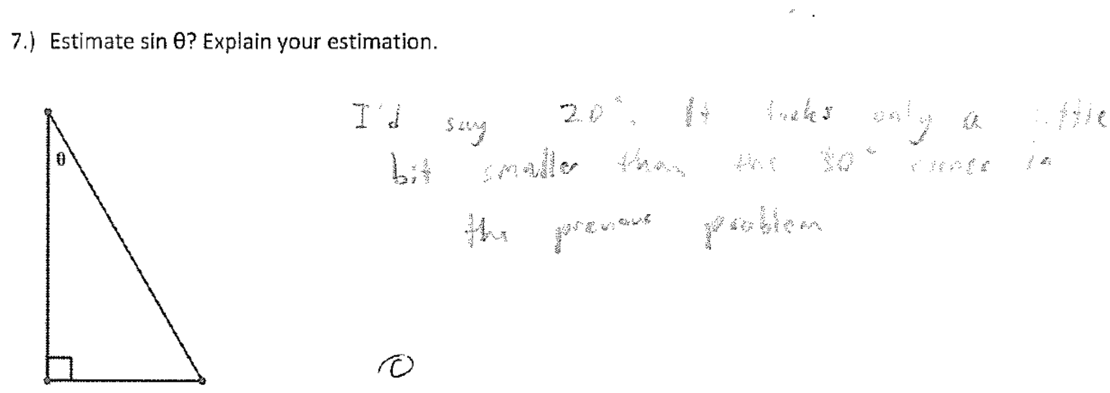
This student scored two points for this question. The student was able to use prior knowledge about common angles to estimate the trigonometric functions of a non-typical angle. The student was able to see that the relationship between 30 degrees and 60 degrees and how it affected each trigonometric function, to help them estimate the trigonometric functions of 20 degrees.

Similar to questions one and two, questions six and seven were designed to ask similar questions, but one was a procedural assessment and the other was a conceptual understanding assessment. In question six, students were given a right triangle that had the lengths of each side and were asked what was sine of an angle. On question seven, students were given a right triangle without the lengths of each side and were asked to find sine of an angle. The goal of this question was to see if students could think of the sine as a ratio, or comparison, of the length of two sides of the triangle. Below is a typical example of student work for question six.



The student scored two points for their response. The student was able to use a procedure of recalling the acronym “SOHCAHTOA” to answer the question. “Mnemonic techniques learned by rote may provide connections among ideas that make it easier to perform mathematical operations, but they also may not lead to understanding” (Kilpatrick, Swafford, & Findell, 2001, p.119). A few typical examples of student work from question 7 are below.

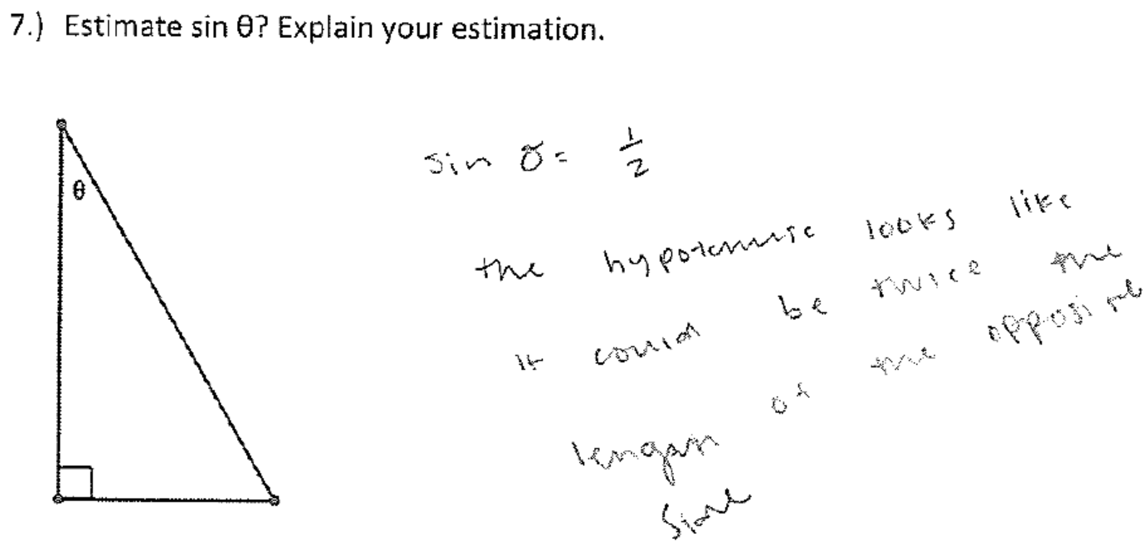




"I'd say 20 degrees. It looks only a little bit smaller than the 30-degree angle in the previous problem."

Both responses to question seven received no points. The first student had a lack of conceptual understanding because he needed numerical values to give an estimation to the problem. The second student showed a lack of conceptual understanding, considering that instead of estimating what the sine was, he solved for the angle measurement instead. Neither student was able to use the fact that the sine ratio is simply the multiplicative comparison of the length of the opposite side to the length of the hypotenuse.

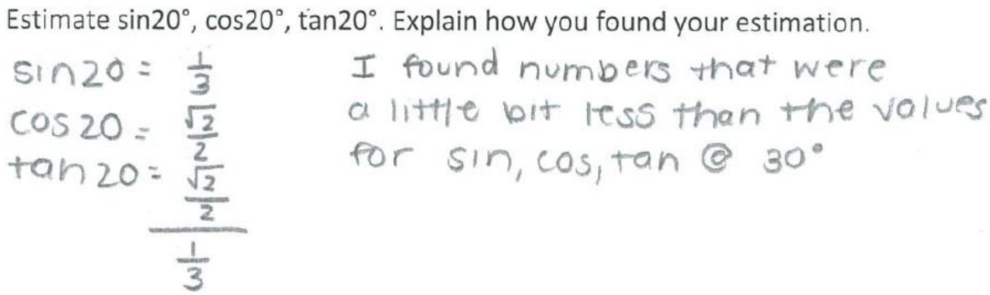
For question six, of the calculus students 84.7 % could answer this question and only 23.3% were able to answer question seven. 62.5% of the trigonometry were able to give an answer to question six and only 21.9% were able to answer question seven. The overall score for question six was 78.8% and the overall score for question seven was 22.9%. Many students showed procedural fluency in their ability to find the sine of a triangle with given side lengths, often through the use of the SOHCAHTOA acronym. But students lacked conceptual understanding considering that some students thought that in order to estimate sin theta, you had to be given side lengths or angle measurements. They were not able to think of the ratio opposite/hypotenuse as a comparison of the length of two sides of the triangle. The student work below is an example that shows a conceptual understanding of the sine as the ratio of two sides of the triangle.



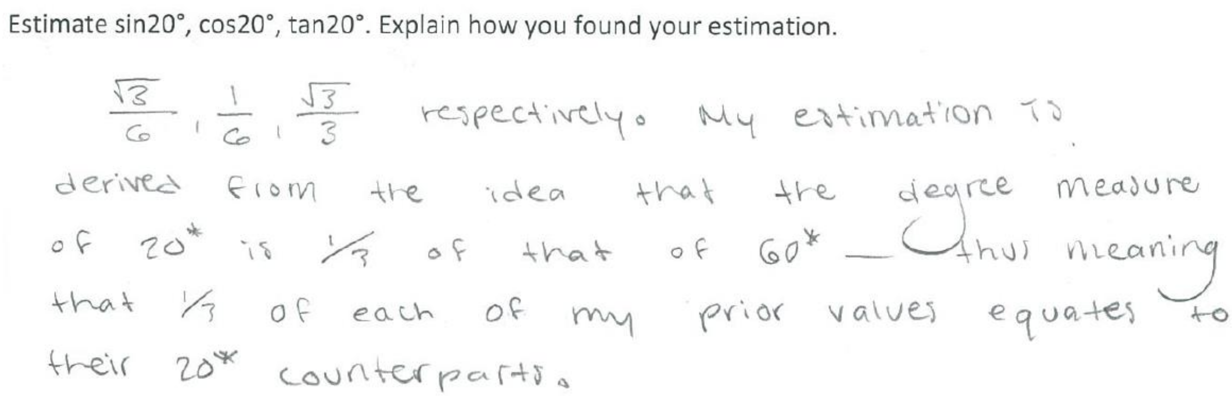
The student score two points for their response. The student was able to visually compare lengths without the need of numbers.

Misconceptions

A lot of misconceptions became obvious as I graded the assessments. One of the misconceptions came out of question 2. This question asked students to find the sine, cosine, and tangent of 20 degrees. Many students believed that there was always a positive correlation between the angle measure and the value of the trigonometric functions. In other words, as the angle gets larger, the trigonometric functions also get larger. Below is an example of student work illustrating this misconception.

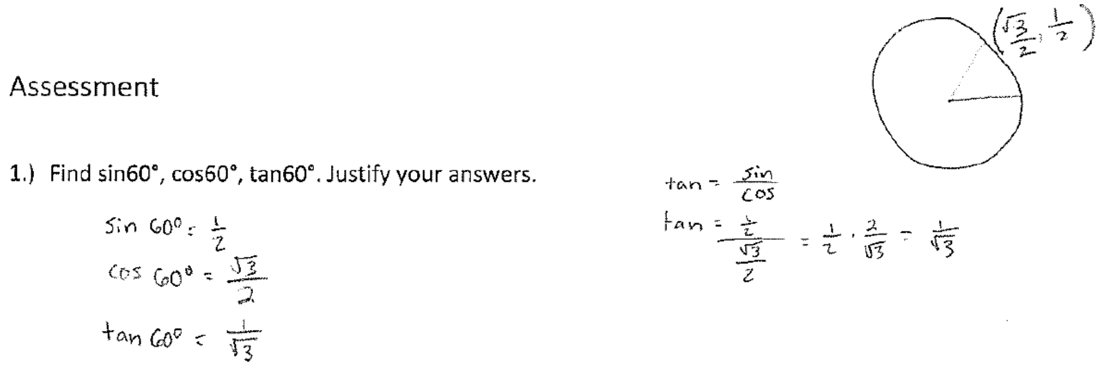


This student estimated all the trigonometric functions of 20 degrees to be slightly less than those at 30 degrees. Although this may give an accurate estimate for the sine and tangent, it will not for the cosine. Moreover, this concept is incorrect for all the trigonometric functions. Another example of this misconception is show in the student work below. It is important to note that this question followed a question asking for the sine, cosine, and tangent of 60 degrees.



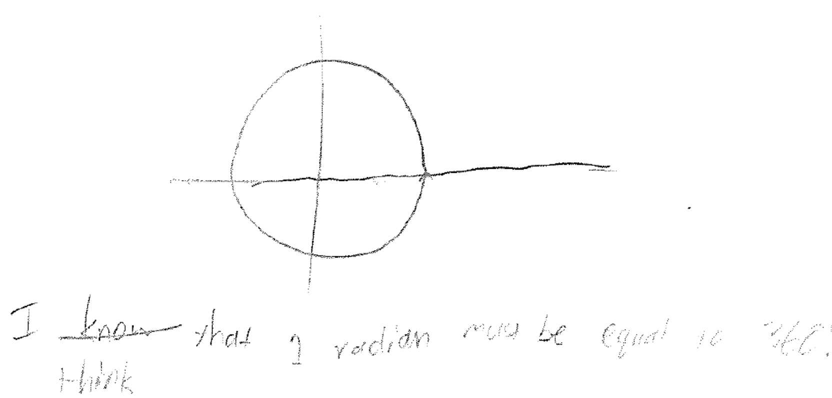
This student used question one to figure out question two, in the way that since 20º is a third of 60º, then the sine, cosine, and tangent would be a third also. In other words, this student believed that $\sin\left(\frac{1}{3}\cdot 60°\right) = \frac{1}{3}(sin60°)$. The overall percentage of students having this misconception is 13.5%, which is partially the reason for the low average of 6.7% on this question.

Questions one and four on the assessment revealed a concept that many students had a partial understanding of. On both questions, many students attempted to use the unit circle to find the desired trigonometric function, but they made mistakes in their ability to correctly recall the memorized unit circle. Some switched the coordinates. For example, in the student work below, the student labelled the point on the unit circle at 60 degrees as $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ instead of $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.



Other students incorrectly identified the x-coordinate as the sine ratio, while some simply wrote incorrect values for the coordinates. 24% of students made this kind of mistake on #1 and 4% on #4. “Learning with understanding is more powerful than simply memorizing because the organization improves retention, promotes fluency, and facilitates learning related material.

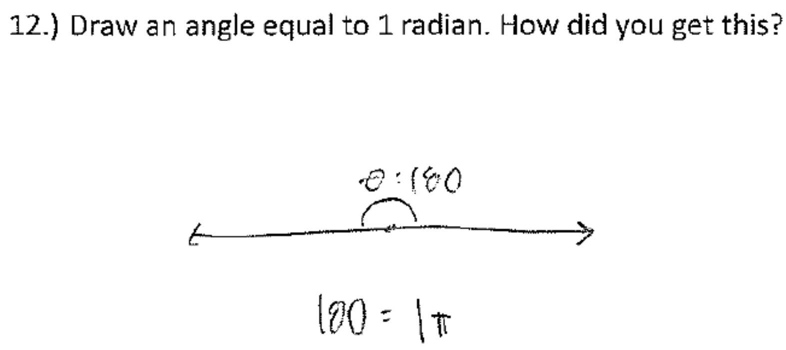
Question twelve on the assessment revealed several misconceptions. Question twelve asked students to draw an angle equal to one radian. One misconception that students showed was that they believe that the whole unit circle was equal to one radian. Below is an example of this misconception.



“I know that 1 radian must be equal to 360 degrees.”

6.7% of students had this misconception.

Another misconception revealed in #12 was that some students thought that 1π radian was equal to 1 radian. The student work below is an example of this misconception.



28.3% of students had this misconception.

These misconceptions contributed to the low overall score of a 12.1% on #12. Another contributor to this low score was the fact that only 3% of students mentioned the length of the radius to conceptualize the angle measured in radians. Below is an example of student work that scored a 2.

Conclusions/Implications

Overall, students have a better procedural understanding of basic trigonometry topics than conceptual understanding. Even though this is the case, the overall procedural fluency score was not that high as I expected. As stated earlier, one reason for this is the fact that students had trouble memorizing formulas correctly. Also, some students did not make connections among graphs to help solve problems. Many of students lack conceptual understanding, with the lack of understanding, students are unable to expand their knowledge of the unit circle and commonly memorized triangles to other less common situations. In other words, students are unable to see relationships among parts of a triangle from a well-known triangle and a non-typical triangle. One reason that this is true, considering that some students have misconceptions/

misunderstandings among trigonometry concepts. For example, as stated and showed previous, students have misconceptions with angles and trigonometric functions. Students believe that there is a positive correlation between angles and trigonometric functions, which is not true. Another example would be that students have trouble understanding the difference between a radian and pi radians. Students believe that they are the same.

I am contributing to the research on common misconceptions that students have in trigonometry. This is helpful for the teachers of trigonometry in increasing their knowledge of students, and in turn increasing their student's mathematical proficiency. By teachers having knowledge of misconceptions that a typical student has, teachers can come up with strategies to help correct the misunderstanding among various trigonometry concepts. "Research suggest that it is vital for students to have an conceptual understanding in order to develop procedural fluency. This suggest that teachers should teach concepts first and let the procedures be developed from that understanding" (Kilpatrick, Swafford, & Findell, 2001, p.369).

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