Unique Visualizations: Exploring Symmetries with Complex-Valued Functions and Group Theory

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Abstract

Symmetrical patterns are present in many areas such as: architecture, art, music, and mathematics. The connection between math and art has been known for thousands of years. Using Fourier analysis, we construct rosette, wallpaper, color-reversing wallpaper and color-turning wallpaper functions to generate symmetry groups. In addition, we create unique visualizations of these functions with the help of the domain-coloring algorithm and a software.

1. Introduction

Throughout history people have enjoyed creating artwork with patterns. Symmetrical patterns, very pleasing to the eye, can be observed in many areas such as: architecture, art, music, and mathematics. Motivated by the connection between math and symmetry, this paper explores various symmetry groups and some constructions of functions that generate these. Using Fourier analysis we will construct rosette, wallpaper, color-reversing wallpaper and color-turning wallpaper functions. With the help of a software developed by Dr. Frank Farris and a technique known as domain coloring we graph these functions to produce unique pieces of art. After the Preliminaries section, where we introduced some definitions and results that are used throughout the paper, we follow with sections on rosettes, frieze and wallpaper patterns. In these sections we mention some of the properties of each group. These are followed by the section Exploring Symmetries with Complex-Valued Functions where we discuss domain coloring, rosette functions, wallpaper functions, color-reversing wallpaper functions and color-turning wallpaper functions. We also present here how we visualize this functions. We end the paper with a conclusion section.

2. Preliminaries

In this section we list some definitions and results that we use in this paper.
Theorem 2.1[5] Let G be a group and a be any element in G. Then the set

$$\langle a \rangle = \{a^k : k \in \mathbb{Z}\}$$

is a subgroup of G.
Definition 2.2[5] Let G be a group. For $a \in G$ we call $\langle a \rangle$ the *cyclic subgroup* generated by a.
Definition 2.3[5] If G is a group and G contains an element a such that $G = \langle a \rangle$ then G is a cyclic group.
Definition 2.4[5] The *pth dihedral group*, denoted by D_p, is the group of rigid motions of a regular p-gon.
Definition 2.5 A *symmetry group* is a collection of all symmetries of a plane figure.

Definition 2.6 A *mirror symmetry* is a symmetry with respect the reflection.
Definition 2.7.6 An object has *rotational symmetry* if it is invariant under rotations.

3. Rosettes and Frieze Patterns

In this section we describe the rosettes and frieze patterns.

Definition 3.1 [4] A *rosette* is a pattern with symmetry group C_p or D_p, where C_p denotes a cyclic group of order p and D_p denotes a dihedral group of order p.

A rosette with only rotational symmetry will generate a cyclic group C_p, and a rosette that has both a rotational symmetry and a mirror symmetry will generate a dihedral group D_p.

Another type of symmetrical pattern is the *frieze pattern*. A frieze pattern is a figure with one direction of translation symmetry. They are also known as "symmetrical strips". Figure 1 demonstrates a few examples of frieze patterns in architecture.

Figure 1: Frieze Patterns in Architecture (Basaralu Temple)

It has been proved that there only seven number of frieze patterns [9]. These patterns can be constructed from five rigid transformations.

Definition 3.2 [9] A *rigid transformation* is a motion that preserves distance between points.

The rigid transformations used to obtain the frieze patterns are: translations(**T**), rotations(**R**), vertical reflections (**V**), horizontal reflections (**H**) and glide reflections (**G**). The most basic of these transformations is a translation, which is the sliding of a tile, either vertical or horizontally but only in one direction. By a rotation we mean that the tile is rotated 180°. Two rotations would bring the tile back to the original position. A vertical reflection is a mirror image of the tile across the y-axis. Similar to a vertical reflection is a horizontal reflection except the reflection occurs across the x-axis. Finally, a glide reflection is a translation followed by a reflection in the same direction as the translation. Figure 2 demonstrates the five rigid transformations.

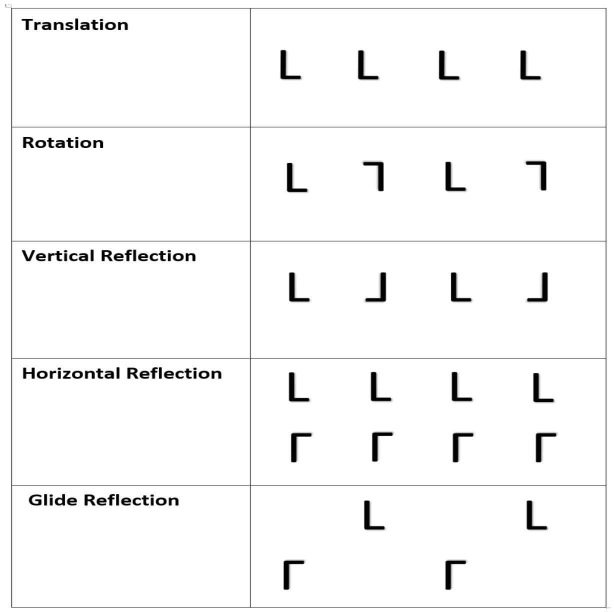

Figure 2: The Five Rigid Transformations

Using these five rigid transformations we get the following seven distinct frieze patterns. The first frieze pattern is generated only by translations (**F1**), the second frieze pattern is constructed with only translations and rotations (**F2**), the third one only by translations and vertical reflections (**F3**), the fourth is generated by translations and glide reflections (**F4**), the fifth is generated only by translations, horizontal reflections, and glide reflections (**F5**), the sixth only by translations, rotations, vertical reflections, and glide reflections (**F6**), and the seventh pattern is generated by translations, vertical reflections, and horizontal reflections (**F7**). For a visual representation of the seven frieze patterns refer to Figure 3.

When we analyze the frieze patterns we can identify two types of groups: the dihedral and the cyclic groups. These groups are the only finite symmetry groups in $\mathbb{R}^2$ [1].

The frieze group F1 is generated by a translation and is infinitely cyclic. Group $F1$ has the form,

$$F1 = \{\tau^n | n \in \mathbb{Z}\}, \text{ where } \tau \text{ is a translation.}$$

The frieze group F2 is generated by a glide reflection and is also infinitely cyclic. This group creates the **TG** frieze pattern.

$$F2 = \{\gamma^n | n \in \mathbb{Z}\}, \text{ where } \gamma \text{ is a glide reflection.}$$

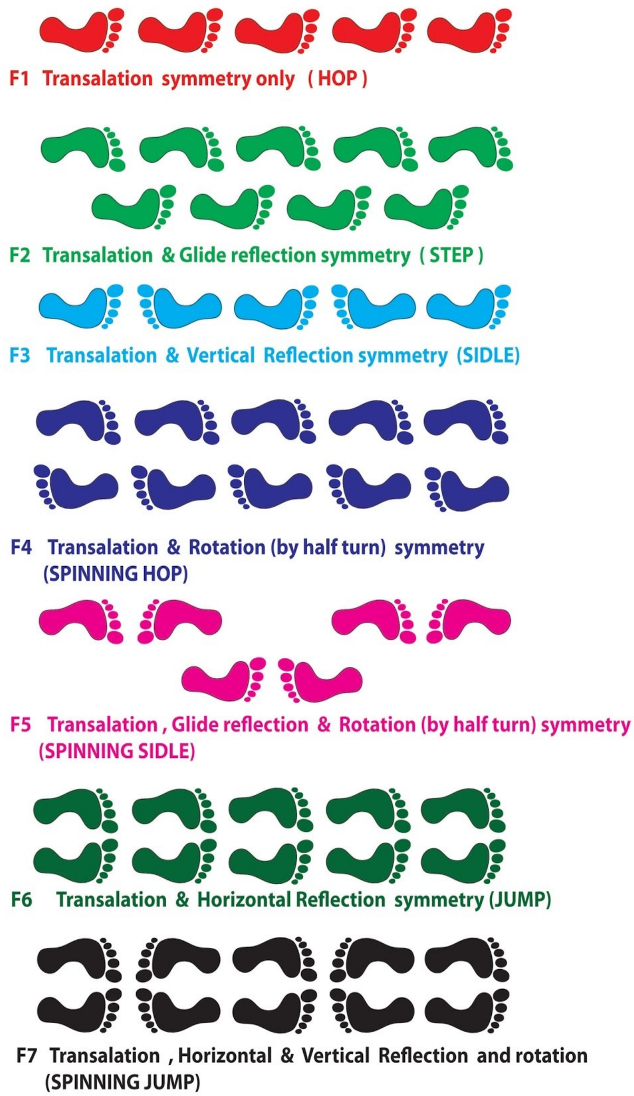

Figure 3: The Seven Frieze Patterns (Conway, 2008)

The frieze group F3 is generated by a translation and a vertical reflection. It is an infinite dihedral group that produces frieze pattern **TV**

$F3 = \{\tau^n\alpha^m | n \in \mathbb{Z}, m = 0 \text{ or } m = 1\}$, where τ is a translation and α is a vertical reflection.

The frieze group F4 is generated by a translation and rotation and is an infinite dihedral group that produces **TR**.

$$F4 = \{\tau^n \rho^m | n \in \mathbb{Z}, m = 0 \text{ or } m = 1\}, \text{ where } \tau \text{ is a translation and } \rho \text{ is a rotation.}$$

The frieze group F5 is generated by a glide reflection and a rotation. It is an infinite dihedral group that creates the pattern **TVRG**. The rotation points are midway between the vertical reflection axis.

$$F5 = \{\gamma^n \rho^m | n \in \mathbb{Z}, m = 0 \text{ or } m = 1\}, \text{ where } \gamma \text{ is a glide reflection and } \rho \text{ is a rotation.}$$

The frieze group F6 is generated by a translation and horizontal reflection and is an abelian group. This group forms the frieze pattern **THG** and is invariant under glide reflection.

$$F6 = \{\tau^n \beta^m | n \in \mathbb{Z}, m = 0 \text{ or } m = 1\}, \text{ where } \tau \text{ is a translation and } \beta \text{ is a horizontal reflection.}$$

Lastly, the group F7 is generated by a translation, a vertical reflection and a horizontal reflection, and it generates the frieze pattern **TVHRG** [4].

$$F7 = \{\tau^n \beta^m \alpha^k | n \in \mathbb{Z}, m = 0 \text{ or } m = 1, k = 0 \text{ or } k = 1\},$$

where τ is a translation, β is a horizontal reflection and α is a horizontal reflection.

4. Wallpaper Patterns

Another type of pattern to consider when discussing symmetry is a wallpaper pattern.

Definition 4.1[7] Wallpaper patterns are repeating patterns that fill a two-dimensional plane.

There are seventeen wallpaper groups [7]. Russian mathematician Evgraf Fedorov proved this fact in the late 19^{th} century. However, it was not until George Polya published a paper in the 1920s that related wallpaper patterns and crystal structure that the classification of the seventeen groups became popular.
After Polya published his paper, a Dutch graphic artist, M.C. Escher read the paper and began to produce tessellations that covered the plane. Though Escher did not know all the mathematical concepts associated with the seventeen wallpaper patterns, he was able to create many of the wallpaper groups in his artwork.

In order to classify the seventeen wallpaper groups we will use the crystallographic notation. Each name can contain up to four symbols. The first letter is either a p, for primitive cell or c for centered. The next is a number which described the order of the rotational symmetry (1, 2, 3, 4, 6) where a rotation of order 1 denotes the identity. The third element will be either an *m, g or 1.* The m denotes a reflection perpendicular to the x-axis, g is a glide reflection perpendicular to the x-axis and 1 means there are none. Similarly, the fourth symbol represents a glide reflection or a reflection about an angle which is determined by the order of the rotational symmetry. Each group has a lattice associated with it. These are: parallelogram, rhombus, square, rectangle and hexagon.

The wallpaper group **p1** is the most basic group. It contains only translations and the lattice is the parallelogram. The group **p2** also has a parallelogram lattice but differs from **p1** because **p2** also contains rotations of 180°.

The symmetry groups **pm** and **pg** both have a rectangular lattice. The group **pm** contains reflections, and **pg** has glide reflections.
There are three more groups that have a rectangular lattice. The first one, the group **pmm** has perpendicular axes of reflection. The second one, **pmg** contains a reflection and rotation of 180°. Lastly, the group **pgg** has glide reflections and half-turns.
There are only two wallpaper groups with a rhombic lattice: **cm** and **cmm**. The wallpaper group **cm** contains reflections and glide reflections, and the group **cmm** has perpendicular reflection axes and rotations of 180°.
Moreover, three wallpaper groups have a square as their lattice. They are: **p4**, **p4m**, and **p4g**. The first group has rotations of 90° and 180°. The group **p4m** is similar to **p4**, but **p4m** has reflections. In this last group the rotation centers are on the reflection axes. Finally **p4g** differs from **p4m** because the rotation centers are not on the reflection axes.
The last five groups each have a hexagonal lattice. They are **p3**, **p31m**, **p3m1**, **p6**, and **p6m**. The groups **p3**, **p31m**, and **p3m1** all have rotations of 120°. In symmetry group **p31m** some of the centers of rotation are on the reflection axes and some are not. Whereas, in **p3m1**, all of the centers of rotation are on the reflection axes. Furthermore, the groups **p6** and **p6m** each have rotations of 180°, 120°, and 60°. The Group **p6** contains no reflections but **p6m** does, and the reflection axes meet the centers of rotations. See Figure 5 for a representation of the seventeen wallpaper groups.

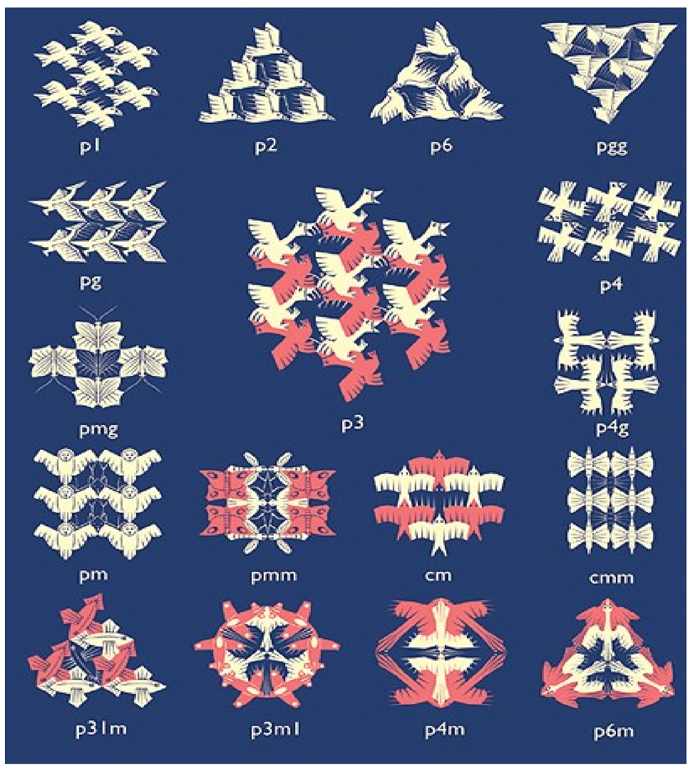

Figure 4: The 17 Wallpaper Groups (Washington Univ., 2009)

5. Exploring Symmetries with Complex-Valued Functions

In this section we describe a few example of how we construct complex-valued functions that create symmetry groups. We use the same idea that Farris described in his book [1]. For this we need to introduce two tools: a family of functions that have group symmetries and the domain coloring.

In order to find complex-valued functions that have group symmetries we use an important theorem from Fourier analysis:
Theorem 5.1[1] If f is a 2π-periodic, complex-valued function of t whose derivative is continuous everywhere, then the Fourier series for f converges to $f(t)$ for every t, and this allow us to write

$$f(t) = \sum_{-\infty}^{\infty} a_n e^{int},$$

where

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt.$$

Analyzing the power series from Theorem 4.1, we observe that this can be written as the restriction of a function to the unit circle. Thus,

$$f(t) = \sum_{-\infty}^{\infty} a_n r^n e^{int} = \sum_{-\infty}^{\infty} a_n z^n.$$

However, we see that negative values of n will make the function undefined at the origin. To ensure we do not have these singularities at the origin, we write it using $\bar{z}$. This gives:

$$f(t) = \sum_{n \geq 0}^{\infty} a_n z^n + \sum_{n>0}^{\infty} a_{-n} \bar{z}^n.$$

This function suggests that for our symmetry study we need to consider the space of functions:

$$\mathcal{F} = \{\sum a_{n,m} z^n \bar{z}^m | a_{n,m} \in \mathbb{C}, n.m \in \mathbb{Z}\}.$$

Before we talk about criteria of finding functions with certain group symmetries, we need to find a way to graph complex-value functions.

A complex-valued function would need a four-dimensional space to visualize, which is not quite possible. However, there is another method called *domain coloring* which is a phrase coined by Frank Farris.

In [4] Farris defines domain coloring as the process of assigning each point on the complex plane a color. Then to view a function f we color each point z in the domain of f with the color corresponding to $f(z)$.

By implementing this algorithm we will be able to visualize the complex functions that we construct in this section.
We will look at an example of domain coloring. We consider the function:

$$f(z) = \frac{z^3 - 1}{z^3 + 1}.$$

We observe that this function has three poles and three zeros. For this example of domain coloring we use a traditional color wheel which can be seen on the left side of the figure below. The white at the center represents the origin, the red is 1, and the black exterior represents very large values. On the right side is what we obtain when we use this color wheel with our function. The three white spots represents the three zeros of f and the the black spots represents the poles of f.

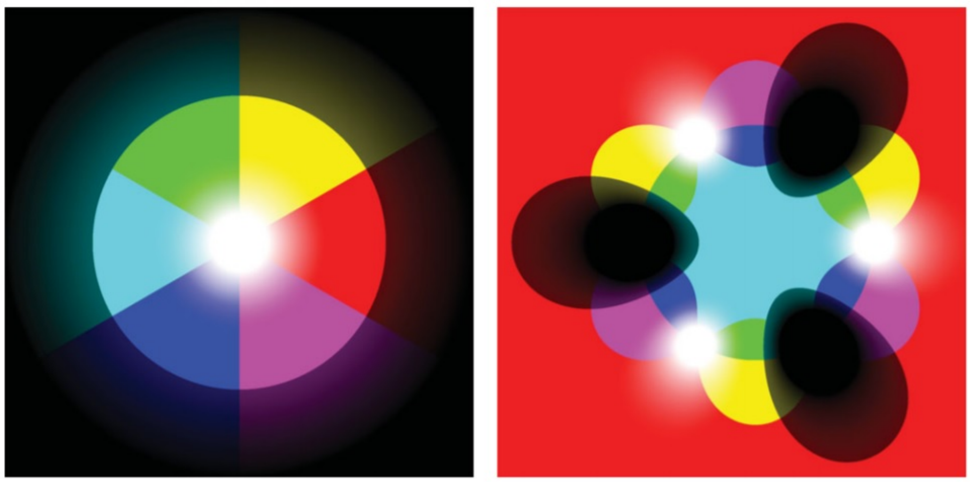

Figure 5: The domain coloring of $f(z) = \frac{z^3-1}{z^3+1}$. (Farris, 2013)

The example above used a traditional color wheel. However, this same process can be applied using photographs as color wheels as we will see later in this section.
Now that we have our tools: the space of functions and the domain coloring, we can talk about how we construct rosette functions, wallpaper functions, color-reversing wallpaper functions and color-turning wallpaper functions.

Rosette Functions

Definition 5.2 [4] A *rosette function with p-fold symmetry* is any function $f \colon \mathbb{C} \to \mathbb{C}$ that is invariant under rotation through an angle of $2\pi/p$.

Theorem 5.3 [4] If in the sum

$$f(z) = \sum a_{n,m} z^n \bar{z}^m$$

we have $a_{n.m} = 0$ unless $n \equiv m$ (mod p), then f is invariant under rotation through an angle of $\frac{2\pi}{p}$. Thus f is a rosette with p-fold symmetry.

Using this theorem we can create any rosette by choosing our n and m to satisfy the conditions which in turn will produce a rosette. Figure 6 represents the function:

$$f(z) = z^6 \bar{z}^{-1} + z^{-3} \bar{z}^4$$

which was graphed using the software created by Dr. Frank Farris, and the color wheel on the left. Note that in our function we have $n \equiv m$ (mod 7). Hence our rosette will have 7 fold symmetry which can be seen in Figure 6.

Figure 6: *Beneath A Jellyfish*

Wallpaper Functions

Definition 5.4 [4] A *wallpaper function* is a function $f : \mathbb{C} \longrightarrow \mathbb{C}$ such that f has two translational symmetries:

$$f(z + \omega_1) = f(z + \omega_2) = f(z)$$

and no translational symmetries by translations smaller than ω_1 or ω_2.

We note that a wallpaper function might have additional symmetries. From Theorem 4 in [4] we have that the symmetries of a function from a group. In our case, the group of a wallpaper function will be isomorphic to one of the 17 wallpaper groups.
In order to construct such a wallpaper function we start with the wallpaper waves defined by Farris in [3]. These waves have the form:

$$E_{n,m}(z) = e^{2\pi i(nX+mY)}$$

where X and Y are the lattice coordinates.

Using Theorem 4.1, we can write our wallpaper function as:

$$f(z) = \sum_{n,m} a_{n,m} E_{n,m} = \sum_{n,m} a_{n,m} e^{2\pi i(nX+mY)}.$$

Now that we have our wallpaper functions, we can construct functions that have wallpaper groups as their symmetric groups. We are going to present one example, namely a function that has its symmetry group isometric to the wallpaper group **p3**. For this we need to introduce the notion of the average of a function over a group.

Definition 5.5[3] Suppose G is a finite group of transformations of the complex plane with n elements and $f(z)$ is a function defined on (some domain in) the complex plane. The *average of f over G* is:

$$\hat{f}(z) = \frac{1}{n} \sum_{g \in G} f(g(z))$$

For example, to produce a 3-fold symmetry we take a simple plane wave

$$f(z) = f(x+iy) = e^{iy}$$

and the 120° rotation function $\rho_3(z) = e^{(2\pi i)/3} z$. We compute the average of f over the cyclic group

$$C_3 = \left\{\tau, \rho_3, \rho_3^2\right\}.$$

Thus we have that the average of f over C_3 is,

$$\hat{f} = \frac{1}{3}(e^{2\pi iy} + e^{2\pi i(\sqrt{3}x-y)/2} + e^{2\pi i(-\sqrt{3}x-y)/2}).$$

If we now apply the averaging technique to $E_{n,m}$, we obtain:

$$W_{m,n}(z) = (e^{2\pi i(nX+mY)} + e^{2\pi i(mX-(n+m)Y)} + e^{2\pi i((-n+m)X+nY)})/3.$$

Consequently, we can define the function

$$f(z) = \sum_{n,m \in \mathbb{Z}} a_{n,m} W_{n,m}(z),$$

that has the wallpaper group **p3** as its symmetry group . If we let n and m be arbitrary integers we can construct infinitely many wallpaper functions of this type. One particular example is represented in Figure 7, where the left side is the color wheel and the right side is the visualization of the function using the domain coloring and Farris's software.

$$f(z) = W_{1,2} + W_{2,1} + W_{0,1}$$

Figure 7: *Butterfly Feeder*

Color-Reversing Wallpaper Functions

Next we are going to describe an example of how we can construct a function with color-reversing symmetry. Before that we are going to mention a few properties related to the color-reversing symmetry of a function.

Definition 5.6 [4] If β is an isometry of the complex plane and $f(z)$ is any function, then β is a color-reversing symmetry (antisymmetry) of f if

$$f(\beta(z)) = -f(z).$$

Definition 5.7 [4] If f is a function then set of symmetries and color-reversing symmetries of f form a group called the **color group** of f.

Proposition 5.8 [4] If β is any color-reversing symmetry of a function f, then β^2 is a symmetry of f. If α is any symmetry of f, then the composition of $\alpha\beta$ and $\beta\alpha$ are color-reversing symmetries. Moreover, if β is a single color-reversing symmetry of a function f whose symmetry group is G, then every color-reversing symmetry of f has the form $\beta\alpha$ for some $a \in G$

Proof. Let β be a color-reversing symmetry and α be a symmetry of a function f. Note that by the definition of color-reversing symmetry,

$$f(\beta^2(z)) = -f(\beta(z)) = f(z).$$

Since $f(\beta^2(z)) = f(z)$, then β^2 is a symmetry of f. Moreover,

$$f(\beta(\alpha(z))) = f(\beta(z)) = -f(z).$$

Similarly,

$$f(\alpha(\beta(z))) = -f(\alpha(z)) = -f(z).$$

Thus we have that the composition of $\alpha\beta$ and $\beta\alpha$ are color-reversing symmetries. Observe that β^{-1} is a color-reversing symmetry whenever β is. Note that for any color-reversing symmetry $\beta^{'}$, the composition $\beta^{-1}\beta^{'}$ is an actual symmetry, $\alpha \in G$. Consequently $\beta^{'} = \beta\alpha$. Hence, we see that β^2 is a symmetry of f, $\alpha\beta$ and $\beta\alpha$ are color-reversing symmetries, and every color-reversing symmetry of f has the form $\beta\alpha$ for some $\alpha \in G$.

$\square$

Definition 5.9 [4] H is a normal subgroup of a group G if H is a subgroup of G and

$$ghg^{-1} \in H, \text{ whenever } h \in H \text{ and } g \in G.$$

Theorem 5.10 [4] The symmetry group of a function f is a normal subgroup of the color group of f.

Proof. Let G denote the color group of a function f and H denote the symmetry group of f. Note that H is a subgroup of G, since the symmetries of f form a group using the same operation of composition as G.

We will prove that H is a normal subgroup of G, that is $ghg^{-1} \in H$ for every $h \in H$ and $g \in G$. Let $g \in G$. Then g can either be a symmetry or a color-reversing symmetry.
Let g be a symmetry of G and h be an element of H. Since $g \in G$ we have that the inverse of g exists. We will call it g^{-1}. Then, we have

$$f(ghg^{-1}(z)) = f(g(h(g^{-1}(z)))) = f(g(h(z))) = f(g(z)) = f(z).$$

Thus $ghg^{-1} \in H$
We now consider the case when g is a color-reversing symmetry. By definition, we have that $f(g(z)) = -f(z)$. It follows that

$$f(ghg^{-1}(z)) = f(g(h(g^{-1}(z)))) = -f(g(h(z))) = -f(g(z)) = f(z).$$

Hence $ghg^{-1} \in H$.

Consequently, since $ghg^{-1} \in H$ for every $h \in H$ and $g \in G$ we have that the symmetry group of a f is a normal subgroup of the color group of f.

$\square$

Let G be the color group of a function f and H the symmetry group of f. Since H is a normal subgroup of G, then classifying the pattern types G/H amounts in finding all the wallpaper groups which are normal subgroups of another. It has been proved that there are 46 of such type of patterns [3].
We will discuss one example out of the 46 cases, namely the group **p31m/p3**.
For this recall our wallpaper function that has the group of symmetries **p3**:

$$f(z) = \sum_{n,m \in \mathbb{Z}} a_{n,m} W_{n,m}(z)$$

where,

$$W_{m,n}(z) = (e^{2\pi i(nX+mY)} + e^{2\pi i(mX-(n+m)Y)} + e^{2\pi i((-n+m)X+nY)})/3$$

In order to create color-reversing symmetries for this function we need to impose some conditions on the coefficients $a_{n,m}$. These conditions that are found by Farris in [4] are:

$$a_{n,m} = a_{m,(-n+m)}$$

$$a_{n,m} = a_{-(n,m),n}$$

$$a_{n,m} = -a_{m,n}$$

An example of this case is the following function:

$$f(z) = W_{2,2} + W_{2,-4} + W_{-4,2} + W_{-2,-2} + W_{3,3}$$

In order to visualize this function using the domain coloring technique and Farris's software we need a special color wheel. This color wheel can be a picture which has a clear divide of the negative and positive values of one of the axis of coordinates. We can see an example of this special wheel on the left hand side of the Figure 8, on the right hand side is the graph of f. We note here that when we make a 120° turn, the colors are reversed.

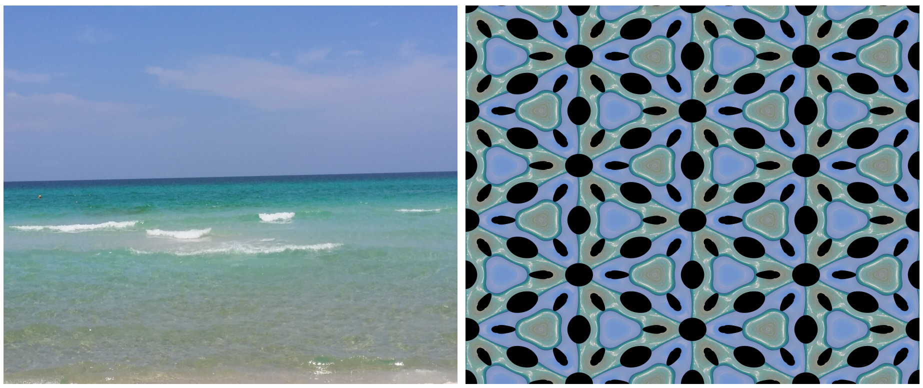

Figure 8: *Snow-cean*

Color-Turning Wallpaper Functions

Definition 5.11 [1] We say α is a p-fold *color-turning symmetry* (of type k) of a function $f(z)$ on the plane, if

$$f(\alpha(z)) = \omega_p^k f(z) \text{ for all } z \in \mathbb{Z},$$

where

$$\omega_p = e^{(2\pi i)/p}.$$

Much like how we created color-reversing wallpaper functions we can create color-turning function.s The color-turning wallpaper functions also require a special color wheel which changes color at the same angle as the turn denoted in the function. For our example we will consider a color-turning function with pattern type **p31m/p3m1** which has rotations of 120°. Thus we start with,

$$W_{m,n}(z) = (e^{2\pi i(nX+mY)} + e^{2\pi i(mX-(n+m)Y)} + e^{2\pi i((-n+m)X+nY)})/3.$$

Using our definition of 3-fold color-turning symmetry on $W_{m,n}(z)$, we obtain:

$$CT_{m,n}(z) = (e^{2\pi i(nX+mY)} + e^{(-2\pi i)/3}e^{2\pi i(mX-(n+m)Y)} + e^{(-4\pi i)/3}e^{2\pi i((-n+m)X+nY)})/3$$

Our 3-fold color-turning wallpaper function will have the form:

$$f(z) = \sum_{n,m \in \mathbb{Z}} a_{n,m} CT_{n,m}(z).$$

Figure 10 demonstrates a color-turning wallpaper. On the left is the special color wheel which changes color at every rotation of 120°. We use this color wheel and

$$f(z) = CT_{2,1} + CT_{-1,-2} + 2CT_{-1,1} + 2CT_{-4,4} + CT_{5,4} + CT_{-4,-5}$$

to create the wallpaper on the right side of the Figure 9. In order to create this wallpaper we need to impose some conditions on the coefficients $a_{n,m}$. These conditions found by Farris in [4] are: $m - n \equiv 1 (\text{mod } 3)$ and to obtain the mirror symmetry we must have $a_{n,m} = a_{-m,-n}$

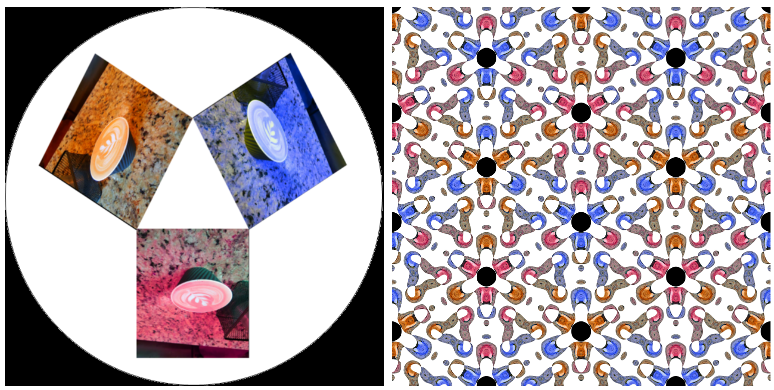

Figure 9: *Coffee Makes the Wheels Go Round*

6. Conclusion

In this paper we have discussed various symmetry patterns: rosette, frieze, and wallpaper. We presented the groups associated with these patterns. Later on, we focused on constructing complex-valued functions which generate a desired symmetry group. In particular, we constructed rosette, wallpaper, color-reversing wallpaper and color-turning wallpaper functions. There are a total of 17 wallpaper groups. The same technique we used to find an example of a function that generates the group **p2**, we can use to create functions which produce the other 16 wallpaper groups. The same idea will work to generate the other 45 color-reversing pattern type groups. For future research, it will be interesting to create functions and apply the domain coloring technique in the study of symmetry in non-Euclidean geometry.

References

[1] Bodner, B. Lynn. The Planar Crystallographic Groups Represented at Alhambra, *Proceedings of Bridges 2013: Mathematics, Music, Art, Architecture, Culture* pp. 225-232, 2013.

[2] Conway, J. H., Burgiel, H., and Goodman-Strauss, C. *The Symmetries of Things.* A.K. Peters, Wellesley, MA, 2008.

[3] Farris, F. A. *Symmetric yet organic: Fourier series as an artist's tool.* Journal of Mathematics and the Arts, 7(2), 64-82, 2013.

[4] Farris, F. A. *Creating symmetry: The artful mathematics of wallpaper patterns.* Princeton University Press. Princeton, NJ, 2015

[5] Gallian, J.A. *Contemporary Abstract Algebra*, Boston, MA, Houghton Mifflin, 1986.

[6] Jablan, S. *Symmetry and Ornament.* Mathematical Institute, Belgrade,Yugoslavia, 2000.

[7] Judson, T. W. *Abstract algebra: Theory and applications.* Orthogonal Publishing, Ann Arbor, MI, 2015.

[8] Schwarzenberger, R.L.E. *The 17 Plane Symmetry Groups*, The Mathematical Gazette, 1974.

[9] Washburn, D. K. and Crowe, D. W. *Symmetries of Culture.* University of Washington Press, 1988.